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FURTHER DEVELOPMENTS IN A NONLINEAR THEORY OF WATER WAVES FOR FINITE AND INFINITE DEPTHS

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CONTENTS

This paper is a companion to an earlier one (Green & Naghdi 1986, *Phil. Trans. R. Soc. Lond.* A **320**, 37–70 (1986)) and deals with certain aspects of a nonlinear water-wave theory and its applications to waters of infinite and finite depths. A new procedure is used to establish a 1–1 correspondence between the lagrangian and eulerian formulations of the integral balance laws of a general thermomechanical theory of directed fluid sheets, as well as their associated jump conditions in the presence of any number of directors. (Such a correspondence between lagrangian and eulerian formulations was previously possible in the special case of a single constrained director.) These results are valid for both compressible and incompressible (not necessarily inviscid) fluids. Applications are then made to special cases of the general theory (including the jump conditions) for incompressible inviscid fluids of infinite depth (with two directors) and of finite depth (with three directors) and the nature of the results are illustrated with particular reference to a wedge-like boat.

1. Introduction

This paper is concerned with further developments in a nonlinear theory of water waves for finite and infinite depths, which are a continuation of previous work (Green & Naghdi 1986). The present developments are dealt with via a *direct* approach on the basis of a model – called *directed* or *Cosserat* surfaces $\mathscr{C}_K$ – comprising a material surface $\mathscr{s}$ with K directors $\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_K$. The earlier basic formulation of such theories, with particular reference to fluid media, was carried out in lagrangian form (Green & Naghdi 1976, 1977, 1979*b*), including that involving thermal effects (1979*a*), and subsequently for special applications the (local) basic equations (in the presence of a single constrained director) were recast in eulerian form. A separate basic general formulation in eulerian form in the context of the purely mechanical theory, which is more convenient for most applications in fluid dynamics, has been effected more recently (Green & Naghdi 1984, 1986). In the latter general eulerian formulations, the basic conservation laws are expressed in terms of integrals over a fixed surface area and its boundary.

For clarity's sake and by way of additional background, we recall from Green & Naghdi (1984, §1) that in the earlier basic formulation of the theories of fluid sheets (or shell-like bodies) the conservation laws were stated by using material surfaces and material derivatives from a lagrangian viewpoint, i.e. time differentiation holding the material point on the surface $\mathscr{s}$ of $\mathscr{C}_K$ fixed. The conversion of the resulting local equations to an eulerian form, in general, posed some difficulty except in the special case of a directed surface with a single constrained director mentioned in the preceeding paragraph. Because of this, a new procedure was effected (Green & Naghdi 1984, appendix), which provided a basis for stating the general conservation laws (in the presence of any number of directors) in an alternative eulerian form.

To describe the contents of this paper, we first note that in an appendix placed at the end, we extend the procedure for conversion from lagrangian to eulerian forms and discuss a method which provides some new ingredients for establishing a 1–1 correspondence between the general integral balance laws in lagrangian form with the corresponding general balance laws in eulerian forms stated earlier (Green & Naghdi 1984, 1986).

With the new procedure of the appendix, we begin the developments of this paper in §2 by stating the general conservation laws (with K directors) for directed sheets in both eulerian and lagrangian forms. This brings the entire general formulation of directed sheets (or Cosserat surfaces) to the same level of common formulation as in the exact three-dimensional theory in which the conversion from lagrangian to eulerian formulations (or vice versa) is straightforward.

The conservation laws in §2 include also those associated with thermal effects, namely the balances of energy and of entropy in both lagrangian and eulerian forms; however, we do not include here the thermodynamical restrictions which arise from considerations of the second law of thermodynamics. Next, with the use of a fixed system of rectangular cartesian coordinates, in §3 we specialize the general theory of §2 to a compressible fluid (not necessarily inviscid), but still retain the general thermodynamical aspects of the developments of §2. General jump conditions with K directors, which accompany the basic equations of §2, are discussed in §4 mainly in eulerian form, but with the limitation to steady flow and with the use of the rectangular cartesian coordinates introduced in §3. Section 4 contains also a detailed discussion and comparison of the jump conditions in both the eulerian and lagrangian forms when the special theory used involves a single director ($K = 1$) which is constrained to be always parallel to a fixed direction. This latter comparison sheds some light on the nature of the jump conditions especially since most of the existing applications (in the presence of a single constrained director) have used lagrangian forms of the jump conditions.

The remainder of the paper is concerned with applications of the developments of §§2–4 to special cases in which only two or three directors are present. Incompressible inviscid fluids of infinite depth (with two directors) and of finite depths (with three directors) are discussed, respectively, in §§5 and 7, and their main features are illustrated with reference to a wedge-like boat in §§6 and 8.

2. Directed sheets

The theory of directed sheets, or Cosserat surfaces $\mathscr{C}_K$, comprising a material surface $\mathscr{s}$ with K directors $\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_K$ is summarized here in a way that gives a direct link between the lagrangian and eulerian points of view instead of the separate treatments given in previous papers. Thermal equations are also included.

Let $\mathscr{s}$, the surface of $\mathscr{C}_K$ in the present configuration at time t, be defined by its position vector $\boldsymbol{r}$ relative to a fixed origin, and let $\theta^\alpha (\alpha = 1, 2)$ be convected (lagrangian) coordinates defining points of the surface. Further, let the K directors be denoted by $\boldsymbol{d}_M (M = 1, 2, \ldots, K)$. Then a motion of the Cosserat surface $\mathscr{C}_K$ is specified by

$$\boldsymbol{d}_0 = \boldsymbol{r} = \boldsymbol{r}(\theta^\alpha, t), \quad \boldsymbol{d}_M = \boldsymbol{d}_M(\theta^\alpha, t). \tag{2.1}$$

Base vectors, unit normal and metric tensors on this surface are denoted by $\boldsymbol{a}_\alpha$, $\boldsymbol{a}^\alpha$, $\boldsymbol{a}_3$, $a_{\alpha\beta}$, $a^{\alpha\beta}$ and are defined in (A 5) of the appendix. The velocity and director velocities are defined by

$$\boldsymbol{v} = \boldsymbol{w}_0 = \dot{\boldsymbol{r}}, \quad \boldsymbol{w}_M = \dot{\boldsymbol{d}}_M, \tag{2.2}$$

where a superposed dot denotes material time differentiation holding θ^α fixed.

Again, let $\bar{\mathscr{s}}$ be a fixed surface in space specified by a position vector $\bar{\boldsymbol{r}}$ which is a function of two curvilinear coordinates $\zeta^\alpha (\alpha = 1, 2)$ on this surface. Base vectors, unit normal and metric tensors on this surface are denoted by $\bar{\boldsymbol{a}}_\alpha$, $\bar{\boldsymbol{a}}^\alpha$, $\bar{\boldsymbol{a}}_3$, $\bar{a}_{\alpha\beta}$, $\bar{a}^{\alpha\beta}$. The surface $\mathscr{s}$ of $\mathscr{C}_K$ in its present configuration at time t, coincides with the fixed surface $\bar{\mathscr{s}}$ and the velocity of points of the moving surfaces $\mathscr{s}$ at this time is denoted by $\bar{\boldsymbol{v}} = \bar{\boldsymbol{v}}(\zeta^\alpha, t)$. Also, in the present configuration at time t when $\mathscr{s}$ coincides with $\bar{\mathscr{s}}$, the director velocities are denoted by $\bar{\boldsymbol{w}}_M = \bar{\boldsymbol{w}}_M(\zeta^\alpha, t)$ with $\bar{\boldsymbol{w}}_0 = \bar{\boldsymbol{v}}(\zeta^\alpha, t)$ and the directors assume the values

$$\bar{\boldsymbol{d}}_M = \bar{\boldsymbol{d}}_M(\zeta^1, \zeta^2) \; (M = 1, 2, \ldots, K), \quad \bar{\boldsymbol{d}}_0 = \bar{\boldsymbol{r}}(\zeta^1, \zeta^2). \tag{2.3}$$

Throughout the paper, we use standard vector and tensor notations with lower case Latin indices (subscripts or superscripts) taking the values 1, 2, 3 and Greek indices the value 1, 2 together with the usual convention for summation over repeated indices (one subscript and one superscript). Here it is necessary to retain the notation of an overbar for quantities related to the surface $\bar{\mathscr{s}}$, in contrast with previous papers.

Consider an arbitrary part $\mathscr{P}$ of the surface $\mathscr{s}$ which coincides with a fixed part $\bar{\mathscr{P}}$ of the surface $\bar{\mathscr{s}}$ at time t and designate the boundary of $\bar{\mathscr{P}}$ by $\partial\bar{\mathscr{P}}$ whose outward unit normal in the surface is

$$\bar{\boldsymbol{\nu}} = \bar{\nu}_\alpha \bar{\boldsymbol{a}}^\alpha = \bar{\nu}^\alpha \bar{\boldsymbol{a}}_\alpha. \tag{2.4}$$

Guided by developments in the appendix we postulate both lagrangian and eulerian forms of the conservation laws for mass, momentum, director momenta and moment of momentum for $\mathscr{C}_K$ as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \rho y_{MN}\,\mathrm{d}\sigma = \frac{\partial}{\partial t}\int_{\bar{\mathscr{P}}} \bar{\rho}\bar{y}_{MN}\,\mathrm{d}\bar{\sigma} + \int_{\partial\bar{\mathscr{P}}} \bar{\rho}\bar{\boldsymbol{v}}_{MN}\cdot\bar{\boldsymbol{\nu}}\,\mathrm{d}\bar{s} - \int_{\bar{\mathscr{P}}} \bar{\rho}(\bar{\boldsymbol{v}}_{MN}+\bar{\boldsymbol{v}}_{NM})\cdot\mathrm{d}\bar{\boldsymbol{\sigma}} = 0 \tag{2.5}$$

for $M, N = 0, 1, 2, \dots, K$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \rho \sum_{M=0}^{K} y_{M0}\boldsymbol{w}_M\,\mathrm{d}\sigma = \frac{\partial}{\partial t}\int_{\bar{\mathscr{P}}} \bar{\rho}\sum_{M=0}^{K} \bar{y}_{M0}\bar{\boldsymbol{w}}_M\,\mathrm{d}\bar{\sigma} + \int_{\partial\bar{\mathscr{P}}} \bar{\rho}\sum_{M=0}^{K} \bar{\boldsymbol{w}}_M\bar{\boldsymbol{v}}_{M0}\cdot\bar{\boldsymbol{\nu}}\,\mathrm{d}\bar{s} = \int_{\bar{\mathscr{P}}} \bar{\rho}\boldsymbol{f}\,\mathrm{d}\bar{\sigma} + \int_{\partial\bar{\mathscr{P}}} \boldsymbol{n}\,\mathrm{d}\bar{s}, \tag{2.6}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \rho \sum_{M=0}^{K} y_{MN}\boldsymbol{w}_M\,\mathrm{d}\sigma &= \frac{\partial}{\partial t}\int_{\bar{\mathscr{P}}} \bar{\rho}\sum_{M=0}^{K} \bar{y}_{MN}\bar{\boldsymbol{w}}_M\,\mathrm{d}\bar{\sigma} + \int_{\partial\bar{\mathscr{P}}} \bar{\rho}\sum_{M=0}^{K} \bar{\boldsymbol{w}}_M\bar{\boldsymbol{v}}_{MN}\cdot\bar{\boldsymbol{\nu}}\,\mathrm{d}\bar{s} - \int_{\bar{\mathscr{P}}} \bar{\rho}\sum_{M=0}^{K} \bar{\boldsymbol{w}}_M\bar{\boldsymbol{v}}_{NM}\cdot\mathrm{d}\bar{\boldsymbol{\sigma}}\\ &= \int_{\bar{\mathscr{P}}} (\bar{\rho}\boldsymbol{l}_N - \boldsymbol{k}_N)\,\mathrm{d}\bar{\sigma} + \int_{\partial\bar{\mathscr{P}}} \boldsymbol{m}_N\,\mathrm{d}\bar{s},\end{aligned} \tag{2.7}$$

for $N = 1, 2, \dots, K$ and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \rho \sum_{N=0}^{K}\sum_{M=0}^{K} y_{NM}\boldsymbol{d}_N\times\boldsymbol{w}_M\,\mathrm{d}\sigma &= \frac{\partial}{\partial t}\int_{\bar{\mathscr{P}}} \bar{\rho}\sum_{N=0}^{K}\sum_{M=0}^{K} \bar{y}_{NM}\bar{\boldsymbol{d}}_N\times\bar{\boldsymbol{w}}_M\,\mathrm{d}\bar{\sigma}\\ &\quad + \int_{\partial\bar{\mathscr{P}}} \bar{\rho}\sum_{N=0}^{K}\sum_{M=0}^{K} \bar{\boldsymbol{d}}_N\times\bar{\boldsymbol{w}}_M\bar{\boldsymbol{v}}_{MN}\cdot\bar{\boldsymbol{\nu}}\,\mathrm{d}\bar{s}\\ &= \int_{\bar{\mathscr{P}}} \bar{\rho}\sum_{N=0}^{K} \bar{\boldsymbol{d}}_N\times\boldsymbol{l}_N\,\mathrm{d}\bar{\sigma} + \int_{\partial\bar{\mathscr{P}}} \sum_{N=0}^{K} \bar{\boldsymbol{d}}_N\times\boldsymbol{m}_N\,\mathrm{d}\bar{s},\end{aligned} \tag{2.8}$$

where $\mathrm{d}\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{a}}_3\,\mathrm{d}\bar{\sigma}$, the inertia coefficients y_{MN}, $\bar{y}_{MN}$ are defined by (A 18), and the velocity fields $\bar{\boldsymbol{v}}_{MN}$ may be expressed in terms of the director velocities with the help of (A 6). Also, $\boldsymbol{n} = \boldsymbol{m}_0$ is the force vector, $\boldsymbol{m}_N$ are the director force vectors at the curve $\partial\bar{\mathscr{P}}$, $\boldsymbol{f} = \boldsymbol{l}_0$ is the assigned force vector, $\boldsymbol{l}_N$ are the assigned director force vectors both per unit mass and $\boldsymbol{k}_N$ are the internal director forces per unit area. The assigned field $\boldsymbol{f}$ may be regarded as representing the combined effect of (i) the stress vector on the major surfaces of the body denoted by $\boldsymbol{f}_c$ and (ii) an integrated contribution arising from the three-dimensional body forces acting on the body denoted by $\boldsymbol{f}_b$. A parallel statement holds for the assigned fields $\boldsymbol{l}_N$. We therefore write

$$\boldsymbol{f} = \boldsymbol{f}_c + \boldsymbol{f}_b, \quad \boldsymbol{l}_N = \boldsymbol{l}_{Nc} + \boldsymbol{l}_{Nb}.$$

Balances of entropy and energy for every part $\mathscr{P}$ of the material surface $\mathscr{s}$ which coincides with the fixed surface $\bar{\mathscr{P}}$ at time t are

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \rho\tilde{\eta}_N\,\mathrm{d}\sigma &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \rho\sum_{M=0}^{N} y_{MN}\eta_M\,\mathrm{d}\sigma\\ &= \frac{\partial}{\partial t}\int_{\bar{\mathscr{P}}} \bar{\rho}\sum_{M=0}^{K} \bar{y}_{MN}\bar{\eta}_M\,\mathrm{d}\bar{\sigma} + \int_{\partial\bar{\mathscr{P}}} \bar{\rho}\sum_{M=0}^{K} \bar{\eta}_M\bar{\boldsymbol{v}}_{MN}\cdot\bar{\boldsymbol{\nu}}\,\mathrm{d}\bar{s} - \int_{\bar{\mathscr{P}}} \bar{\rho}\sum_{M=0}^{K} \bar{\eta}_M\bar{\boldsymbol{v}}_{NM}\cdot\mathrm{d}\bar{\boldsymbol{\sigma}}\\ &= \int_{\bar{\mathscr{P}}} \bar{\rho}(s_N + \xi_N)\,\mathrm{d}\bar{\sigma} - \int_{\partial\bar{\mathscr{P}}} k_N\,\mathrm{d}\bar{s} \quad (N = 0, 1, 2, \dots K)\end{aligned} \tag{2.9}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}\left(\epsilon+\frac{1}{2}\sum_{M=0}^{K}\sum_{N=0}^{K}y_{MN}\,\boldsymbol{w}_M\cdot\boldsymbol{w}_N\right)\rho\,\mathrm{d}\sigma=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}\sum_{M=0}^{K}\sum_{N=0}^{K}y_{MN}(\epsilon_{MN}+\tfrac{1}{2}\boldsymbol{w}_M\cdot\boldsymbol{w}_N)\,\rho\,\mathrm{d}\sigma$$
$$=\frac{\partial}{\partial t}\int_{\mathscr{P}}\sum_{M=0}^{K}\sum_{N=0}^{K}\bar{y}_{MN}(\bar{\epsilon}_{MN}+\tfrac{1}{2}\bar{\boldsymbol{w}}_M\cdot\bar{\boldsymbol{w}}_N)\,\bar{\rho}\,\mathrm{d}\bar{\sigma}+\int_{\partial\mathscr{P}}\sum_{M=0}^{K}\sum_{N=0}^{K}(\bar{\epsilon}_{MN}+\tfrac{1}{2}\bar{\boldsymbol{w}}_M\cdot\bar{\boldsymbol{w}}_N)\,\bar{\boldsymbol{v}}_{MN}\cdot\bar{\boldsymbol{v}}\bar{\rho}\,\mathrm{d}s$$
$$=\int_{\mathscr{P}}\sum_{N=0}^{K}(r_N+\boldsymbol{l}_N\cdot\bar{\boldsymbol{w}}_N)\,\bar{\rho}\,\mathrm{d}\bar{\sigma}+\int_{\partial\mathscr{P}}\sum_{N=0}^{K}(\boldsymbol{m}_N\cdot\bar{\boldsymbol{w}}_N-h_N)\,\mathrm{d}\bar{s}. \qquad (2.10)$$

In these equations $\bar{\eta}_N$, $\eta_N(\theta^\alpha, t)=\bar{\eta}_N(\zeta^\alpha, t)$ are the entropy densities, k_N are the entropy fluxes, h_N are the heat fluxes, s_N are the external rates of supply of entropy, r_N are the external rates of supply of heat, ϵ, $\epsilon_{MN}(\theta^\alpha, t)=\bar{\epsilon}_{MN}(\zeta^\alpha, t)$ are internal energy densities. If $\theta_N(N=0, 1, 2, \ldots, K)$ represent the effects of temperature then

$$r_N=\theta_N s_N,\quad h_N=\theta_N k_N\quad (N=0,1,2,\ldots,K;\ N\text{ not summed}). \qquad (2.11)$$

The field equations corresponding to (2.5) are

$$\left.\begin{aligned}\dot{\rho}+\rho\,\mathrm{div}_s\boldsymbol{v}=0,\quad &\rho a^{\frac{1}{2}}=\text{function of }\theta^\alpha\text{ only},\\ y_{00}=1,\quad &y_{MN}=\text{function of }\theta^\alpha\text{ only},\end{aligned}\right\} \qquad (2.12a)$$

or

$$\frac{\partial}{\partial t}(\bar{\rho}\bar{y}_{MN})+\bar{a}^{-\frac{1}{2}}(\bar{\rho}\bar{a}^{\frac{1}{2}}\bar{\boldsymbol{v}}_{MN}\cdot\bar{\boldsymbol{a}}^\alpha)_{,\alpha}-\bar{\rho}(\bar{\boldsymbol{v}}_{MN}+\bar{\boldsymbol{v}}_{NM})\cdot\bar{\boldsymbol{a}}_3=0, \qquad (2.12b)$$

where $(\ )_{,\alpha}=\partial(\ )/\partial\zeta^\alpha$. Similarly, making use of (2.12 b) the local field equations corresponding to (2.6) to (2.8) are

$$\rho\sum_{M=0}^{K}y_{MN}\,\dot{\boldsymbol{w}}_M=\bar{\rho}\sum_{M=0}^{K}\left\{\bar{y}_{MN}\frac{\partial\bar{\boldsymbol{w}}_M}{\partial t}+(\bar{\boldsymbol{v}}_{MN}\cdot\bar{\boldsymbol{a}}^\alpha)\frac{\partial\bar{\boldsymbol{w}}_M}{\partial\zeta^\alpha}+\bar{\boldsymbol{w}}_M\bar{\boldsymbol{v}}_{MN}\cdot\bar{\boldsymbol{a}}_3\right\}$$
$$=\bar{\rho}\boldsymbol{l}_N-\boldsymbol{k}_N+\boldsymbol{M}_N^\alpha|_\alpha \qquad (2.13)$$

for $N=0,1,\ldots,K$, where $\boldsymbol{k}_0=\boldsymbol{0}, \boldsymbol{M}_0^\alpha=\boldsymbol{N}^\alpha$, and

$$\sum_{N=0}^{K}((\bar{\boldsymbol{d}}_N\times\boldsymbol{k}_N+\bar{\boldsymbol{d}}_{N,\alpha}\times\boldsymbol{M}_N^\alpha)=\boldsymbol{0}. \qquad (2.14)$$

A vertical line denotes covariant differentiation with respect to the surface $\bar{\sigma}$. Also

$$\boldsymbol{n}=\boldsymbol{N}^\alpha\nu_\alpha,\quad \boldsymbol{m}_N=\boldsymbol{M}_N^\alpha\nu_\alpha. \qquad (2.15)$$

Again, from (2.12) and (2.13), the field equations corresponding to (2.9) and (2.10) are

$$\rho\dot{\bar{\eta}}_N=\rho\sum_{M=0}^{K}y_{MN}\dot{\eta}_M=\bar{\rho}\sum_{M=0}^{K}\left\{\bar{y}_{MN}\frac{\partial\bar{\eta}_M}{\partial t}+(\bar{\boldsymbol{v}}_{MN}\cdot\bar{\boldsymbol{a}}^\alpha)\frac{\partial\bar{\eta}_M}{\partial\zeta^\alpha}+\bar{\eta}_M\bar{\boldsymbol{v}}_{MN}\cdot\bar{\boldsymbol{a}}_3\right\}$$
$$=\bar{\rho}(s_N+\xi_N)-\mathrm{div}_s\boldsymbol{p}_N$$
$$k_N=\boldsymbol{p}_N\cdot\bar{\boldsymbol{v}},\quad \bar{a}^{\frac{1}{2}}\mathrm{div}_s\boldsymbol{p}_N=(\bar{a}^{\frac{1}{2}}\boldsymbol{p}_N\cdot\bar{\boldsymbol{a}}^\alpha)_{,\alpha}, \qquad (2.16)$$

$$\rho\dot{\epsilon}=\rho\sum_{M=0}^{K}\sum_{N=0}^{K}y_{MN}\dot{\epsilon}_{MN}$$
$$=\bar{\rho}\sum_{M=0}^{K}\sum_{N=0}^{K}\left\{\bar{y}_{MN}\frac{\partial\bar{\epsilon}_{MN}}{\partial t}+(\bar{\boldsymbol{v}}_{MN}\cdot\bar{\boldsymbol{a}}^\alpha)\frac{\partial\bar{\epsilon}_{MN}}{\partial\zeta^\alpha}+\bar{\epsilon}_{MN}(\bar{\boldsymbol{v}}_{MN}+\bar{\boldsymbol{v}}_{NM})\cdot\bar{\boldsymbol{a}}_3\right\}$$
$$=\sum_{N=0}^{K}(\bar{\rho}\theta_N s_N-\boldsymbol{p}_N\cdot\boldsymbol{g}_N-\theta_N\,\mathrm{div}_s\boldsymbol{p}_N)+P, \qquad (2.17)$$

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where
$$\left.\begin{aligned} P &= \sum_{N=0}^{K} (\boldsymbol{k}_N \cdot \bar{\boldsymbol{w}}_N + \boldsymbol{M}_N^\alpha \cdot \bar{\boldsymbol{w}}_{N,\alpha}), \quad \boldsymbol{k}_0 = \boldsymbol{0}, \\ \boldsymbol{g}_N &= \operatorname{grad} \theta_N = \theta_{N,\alpha} \bar{\boldsymbol{a}}^\alpha \end{aligned}\right\} \tag{2.18}$$

and external body forces are eliminated with the help of (2.13). After elimination of external rates of supply of heat by (2.16), (2.17) becomes

$$\begin{aligned} \rho\left(\dot{\psi} + \sum_{N=0}^{K} \tilde{\eta}_N \dot{\theta}_N\right) &= \rho \sum_{M=0}^{K} \sum_{N=0}^{K} y_{MN}(\dot{\psi}_{MN} + \eta_M \dot{\theta}_N) \\ &= \bar{\rho} \sum_{M=0}^{K} \sum_{N=0}^{K} \left\{ \bar{y}_{MN} \left(\frac{\partial \bar{\psi}_{MN}}{\partial t} + \bar{\eta}_M \frac{\partial \theta_N}{\partial t} \right) \right. \\ &\quad \left. + (\bar{\boldsymbol{v}}_{MN} \cdot \bar{\boldsymbol{a}}^\alpha) \left(\frac{\partial \bar{\psi}_{MN}}{\partial \zeta^\alpha} + \bar{\eta}_M \frac{\partial \theta_N}{\partial \zeta^\alpha} \right) + (\bar{\boldsymbol{v}}_{MN} \cdot \bar{\boldsymbol{a}}_3)\,(2\bar{\psi}_{MN} + \bar{\eta}_N \theta_M) \right\} \\ &= -\sum_{N=0}^{K} (\bar{\rho}\theta_N \xi_N + \boldsymbol{p}_N \cdot \boldsymbol{g}_N) + P, \end{aligned} \tag{2.19}$$

where
$$\left.\begin{aligned} \psi &= \epsilon - \sum_{N=0}^{K} \theta_N \tilde{\eta}_N = \sum_{M=0}^{K} \sum_{N=0}^{K} (\epsilon_{MN} - \theta_N \eta_M), \\ \bar{\psi}_{MN} &= \bar{\epsilon}_{MN} - \tfrac{1}{2}(\bar{\eta}_M \theta_N + \bar{\eta}_N \theta_M). \end{aligned}\right\} \tag{2.20}$$

After constitutive equations have been chosen for the response functions

$$\psi (\text{or } \psi_{MN}, \bar{\psi}_{MN}), \quad \tilde{\eta}_N \ (\text{or } \eta_N, \bar{\eta}_N), \quad \theta_N, \xi_N, \boldsymbol{p}_N, \boldsymbol{k}_N, \boldsymbol{M}_N^\alpha, \tag{2.21}$$

then (2.19) is regarded as an identity for all thermomechanical processes.

To complete the theory we need to examine restrictions arising from interpretations of the second law of thermodynamics, but we omit this.

3. Inviscid fluids

The general theory of §2 is applied to the special case of a compressible inviscid fluid before further specialization to particular geometries. We consider a compressible fluid of density ρ^* and choose a fixed system of rectangular cartesian coordinates $\zeta^i = x_i (i = 1, 2, 3)$ with corresponding orthonormal base vectors $\boldsymbol{e}_i$. The fixed surface $\bar{\mathscr{s}}$ is a plane surface $\zeta = x_3$ constant. Then we may set

$$\zeta^\alpha = x_\alpha, \quad \bar{\boldsymbol{a}}_\alpha = \bar{\boldsymbol{a}}^\alpha = \boldsymbol{e}_\alpha, \quad \bar{\boldsymbol{a}}_3 = \boldsymbol{e}_3, \quad \bar{a}_{\alpha\beta} = \bar{a}^{\alpha\beta} = \delta_{\alpha\beta}. \tag{3.1}$$

The moving surface $\mathscr{s}$ coincides with the fixed surface $\bar{\mathscr{s}}$ at time t so that, after operations such as material time derivatives have been carried out, we may set

$$\boldsymbol{a}_\alpha = \bar{\boldsymbol{a}}_\alpha = \boldsymbol{e}_\alpha, \quad y_{MN} = \bar{y}_{MN}, \quad \text{etc.} \tag{3.2}$$

For guidance we briefly examine the problem of inviscid fluids in the usual three dimensional context. The thermodynamic identity which results from eliminating external body forces and external rate of supply of heat from the energy equation is

$$\rho^*(\dot{\psi}^* + \eta^* \dot{\theta}^*) = -\theta^* \xi^* - \boldsymbol{p}^* \cdot \boldsymbol{g}^* + \boldsymbol{T} \cdot \boldsymbol{D}, \tag{3.3}$$

where $\psi^*, \eta^*, \xi^*, \boldsymbol{p}^*, \boldsymbol{T}$, are respectively the Helmholtz function, entropy density, internal rate of production of entropy, entropy flux vector and stress tensor. Also $\boldsymbol{D}$ is the rate of deformation,

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$\theta^*(>0)$ is a measure of temperature and $\boldsymbol{g}^* = \text{grad}^*\,\theta^*$. For an inviscid (compressible) fluid we assume that

$$\psi^*, \eta^*, \xi^*, \boldsymbol{p}^*, \boldsymbol{T}, \theta^* \tag{3.4}$$

are functions of ρ^*, T^* and grad T^*, where T^* is an empirical temperature. Use of (3.3) as an identity then shows that

$$\theta^* = \theta^*(T^*) \tag{3.5}$$

independent of ρ^* and grad* T^* so that, with the additional restriction $\partial\theta^*/\partial T^* > 0$, we may use $\theta^* > 0$ as an absolute temperature for this class of fluid, instead of T^*, the empirical temperature. Then

$$\left.\begin{aligned} \psi^* &= \psi^*(\rho^*, \theta^*), \eta^* = -\partial\psi^*/\partial\theta^*, \\ \boldsymbol{T} &= -p^*\boldsymbol{I}, p^* = \rho^{*2}\,\partial\psi^*/\partial\rho^*, \\ \boldsymbol{p}^* &= \boldsymbol{p}^*(\rho^*, \theta^*, \text{grad}\,\theta^*) \\ &= -k(\rho^*, \theta^*, \boldsymbol{g}^*\cdot\boldsymbol{g}^*)\,\boldsymbol{g}^*. \end{aligned}\right\} \tag{3.6}$$

In deriving the last form for $\boldsymbol{p}^*$ use has been made of invariance conditions under superposed rigid body motions.

Returning to the theory of §2 with choice (3.1) for the fixed reference surface we assume that ρ^*, T^* are represented by

$$\rho^* = \sum_{N=0}^{K} \rho_N \lambda_N(x_M), \quad T^* = \sum_{N=0}^{K} T_N \lambda_N(x_M). \tag{3.7}$$

For inviscid fluids the constitutive functions (2.21) are assumed to depend on

$$\rho_N,\ T_N,\ \text{grad}\ T_N = \boldsymbol{e}_\alpha\, T_{N,\alpha}, \tag{3.8}$$

where T_N are measures of empirical temperatures. Then, by using (2.19) as an identity for all thermomechanical processes, in which external body forces and external rates of supply of entropy are chosen to balance the equations of motion and entropy balance equations, it follows that

$$\theta_N = \theta_N(T_M) \tag{3.9}$$

for $M, N = 0, 1, \ldots, K$. With suitable restrictions we may express T_M as functions of θ_N and replace T_M by θ_N in the constitutive equations. Also ψ (or $\psi_{MN}, \bar{\psi}_{MN}$) depend only on ρ_N, θ_N. In further application of (2.19) it is convenient to use the lagrangian form of the equation involving the first group of terms on the left-hand side. Also, corresponding to the assumptions for ψ, the Helmholtz function ψ^* has the form in (3.6) and, from a result analagous to $(\text{A }23)_1$,

$$\begin{aligned} \rho\dot{\psi} &= \int_{z_1}^{z_2} \rho^*\left(\frac{\partial\psi^*}{\partial\rho^*}\dot{\rho}^* + \frac{\partial\psi^*}{\partial\theta^*}\dot{\theta}^*\right)\mathrm{d}z \\ &= -\sum_{M=0}^{K}\int_{z_1}^{z_2}\rho^{*2}\frac{\partial\psi^*}{\partial\rho^*}(\boldsymbol{w}_{M,\alpha}\cdot\boldsymbol{e}_\alpha\lambda_M + \boldsymbol{w}_M\cdot\boldsymbol{e}_3\lambda'_M)\,\mathrm{d}z + \sum_{M=0}^{K}\int_{z_1}^{z_2}\rho^*\frac{\partial\psi^*}{\partial\theta_M}\dot{\theta}_M\,\mathrm{d}z. \end{aligned} \tag{3.10}$$

Then

$$\rho\sum_{M=0}^{K} y_{MN}\,\eta_M = \rho\bar{\eta}_N = -\frac{\partial}{\partial\theta_N}\int_{z_1}^{z_2}\rho^*\psi^*\,\mathrm{d}z = -\rho\frac{\partial\psi}{\partial\theta_N}, \tag{3.11}$$

$$\boldsymbol{M}_N^\alpha = -\boldsymbol{e}_\alpha\int_{z_1}^{z_2}\rho^{*2}\frac{\partial\psi^*}{\partial\rho^*}\lambda_N\,\mathrm{d}z, \quad \boldsymbol{k}_N = -\boldsymbol{e}_3\int_{z_1}^{z_2}\rho^{*2}\frac{\partial\psi^*}{\partial\rho^*}\lambda'_N\,\mathrm{d}s. \tag{3.12}$$

One case of special interest is that of an ideal gas of constant specific heat c_v for which

$$\psi^* = \bar{R}\theta^* \ln\rho^* + c_v(\theta^* - \theta^* \lg\theta^*), \quad \bar{R} = R/M = c_v(\gamma-1), \tag{3.13}$$

where M is molecular mass and R, γ are constants. Then

$$\rho \sum_{M=0}^{K} y_{MN}\eta_M = \rho\tilde{\eta}_N = -\int_{z_1}^{z_2} (\bar{R}\ln\rho^* - c_v \lg\theta^*)\,\rho^*\lambda_N\,dz, \tag{3.14}$$

$$\left.\begin{aligned} M_N^\alpha &= -e_\alpha \int_{z_1}^{z_2} \bar{R}\rho^*\theta^*\lambda_N\,dz = -e_\alpha \bar{R} \sum_{M=0}^{K} \rho y_{MN}\theta_M, \\ k_N &= -e_3 \int_{z_1}^{z_2} \bar{R}\rho^*\theta^*\lambda'_N\,dz = -e_3 \bar{R} \sum_{M=0}^{K} \rho y'_{MN}\theta_M, \end{aligned}\right\} \tag{3.15}$$

where

$$\rho y'_{MN} = \int_{z_1}^{z_2} \rho^*\lambda_M\lambda'_N\,dz. \tag{3.16}$$

If the entropy η^* is everywhere constant, then $\tilde{\eta}_N$ are constants,

$$\theta^* = \exp(\eta^*/c_v)\,\rho^{*(\gamma-1)} \tag{3.17}$$

and

$$M_N^\alpha = -Ae_\alpha \int_{z_1}^{z_2} \rho^{*\gamma}\lambda_N\,dz, \quad k_N = -Ae_3 \int_{z_1}^{z_2} \rho^{*\gamma}\lambda'_N\,dz, \tag{3.18}$$

where A is a constant. On the other hand, if temperature θ^* is constant everywhere; then θ is constant, $\theta_N = 0 (N = 1, 2, \ldots, K)$ and

$$\left.\begin{aligned} M_N^\alpha &= -Be_\alpha \int_{z_1}^{z_2} \rho^*\lambda_N\,dz = -Be_\alpha \rho y_{0N}, \\ k_N &= -Be_3 \int_{z_1}^{z_2} \rho^*\lambda'_N\,dz = -Be_3 \rho y'_{0N}, \end{aligned}\right\} \tag{3.19}$$

where B is a constant.

We next suppose that the fluid is incompressible with ρ^* constant everywhere and incompressibility conditions of the form

$$\sum_{M=0}^{K} (a_{MN} \mathbf{w}_{M,\alpha}\cdot e_\alpha + b_{MN} \mathbf{w}_M \cdot e_3) = 0 \tag{3.20}$$

for $N = 0, 1, 2, \ldots, K$. These conditions induce constraint responses in the functions M_N^α, k_N of the form

$$M_N^\alpha = -e_\alpha \sum_{M=0}^{K} a_{NM} p_M, \quad k_N = -e_3 \sum_{M=0}^{K} b_{NM} p_M, \tag{3.21}$$

where p_M are arbitrary functions of x_α, t which have also absorbed the contributions from (3.19). The entropies are still given by (3.14) and, from $(3.7)_1$, ρ_0 is constant and $\rho_M = 0 (M = 1, 2, \ldots, K)$.

If, in addition, the temperature is constant everywhere then θ_0 is constant, $\theta_N = 0 (N = 1, 2, \ldots, K)$ and

$$\tilde{\eta}_N = \text{const.}, \quad \boldsymbol{p}_N = \boldsymbol{0}, \quad \xi_N = 0, \quad s_N = 0, \quad \dot{\epsilon} = 0. \tag{3.22}$$

In applications, constraints other than the constraint of incompressibility, are introduced into the theory but these are best considered for each special case.

4. Jump conditions for steady flow

The field equations obtained in §2 are valid only in regions where there are no discontinuities. In obtaining the solutions to certain problems, however, it may be necessary to consider boundaries at which there are discontinuities in both the kinematic and kinetic variables and this requires considerations of jump conditions across such boundaries. Previously, the jump conditions have been obtained with various degrees of generality by Green & Naghdi (1976, 1977), Caulk (1976) and Naghdi & Rubin (1981, 1987) for a theory with one director from the lagrangian form of the integral balance equations. Here, we return to the integral balance equations (2.5)–(2.10) and derive the appropriate jump conditions for K directors from the eulerian forms of these equations. In the interest of simplicity, we restrict our attention only to jump conditions for steady two-dimensional flow in planes perpendicular to the x_2-direction using the special rectangular cartesian coordinate system introduced in §3. Thus, we set

$$\left.\begin{aligned} &\boldsymbol{w}_M = \bar{\boldsymbol{w}}_M = w_{Mi}\,\boldsymbol{e}_i, \quad \boldsymbol{w}_0 = \boldsymbol{v} = \bar{\boldsymbol{v}}, \quad \bar{\boldsymbol{v}}_{MN} = v_{MNi}\,\boldsymbol{e}_i, \\ &\boldsymbol{l}_N = l_{Ni}\,\boldsymbol{e}_i, \quad \boldsymbol{k}_N = k_{Ni}\,\boldsymbol{e}_i, \quad \boldsymbol{m}_N = m_{Ni}\,\boldsymbol{e}_i, \quad \boldsymbol{n} = \boldsymbol{m}_0 = n_i\,\boldsymbol{e}_i, \end{aligned}\right\} \tag{4.1}$$

where

$$w_{M2} = 0, \quad v_{MN2} = 0. \tag{4.2}$$

We suppose that there is a discontinuity at $x_1 = x_0$ and apply the eulerian forms of (2.5)–(2.10) to a strip bounded by $x_1 = x_0 - \delta$ and $x_1 = x_0 + \delta$ and then take the limit as $\delta \to 0$. Because the motion is steady, there is no contribution to the jump condition from the time derivative $\partial/\partial t$ of the surface integrals. The contributions from the line integrals is immediately of the form (4.8) listed below, but there may also be contributions from the remaining surface integrals because their integrands may contain singularities in the region $(x_0 - \delta, x_0 + \delta)$. In the limit as $\delta \to 0$, it follows from (2.5)–(2.7) that

$$[\bar{\rho} v_{MN1}] - \lim_{\delta \to 0} \int_{x_0-\delta}^{x_0+\delta} (v_{MN3} + v_{NM3})\,\mathrm{d}x = 0, \tag{4.3}$$

$$\left[\!\left[\bar{\rho} \sum_{M=0}^{K} \boldsymbol{w}_M v_{M01}\right]\!\right] = [\boldsymbol{n}] + \lim_{\delta \to 0} \int_{x_0-\delta}^{x_0+\delta} \rho \boldsymbol{f}\,\mathrm{d}x, \tag{4.4}$$

$$\left[\!\left[\bar{\rho} \sum_{M=0}^{K} \boldsymbol{w}_M v_{MN1}\right]\!\right] - \lim_{\delta \to 0} \int_{x_0-\delta}^{x_0+\delta} \bar{\rho} \sum_{M=0}^{K} \boldsymbol{w}_M v_{NM3}\,\mathrm{d}x = [\boldsymbol{m}_N] + \lim_{\delta \to 0} \int_{x_0-\delta}^{x_0+\delta} (\bar{\rho} \boldsymbol{l}_N - \boldsymbol{k}_N)\,\mathrm{d}x. \tag{4.5}$$

From the entropy balances (2.9) and energy equation (2.10) the jump conditions are

$$\left[\!\left[\bar{\rho} \sum_{M=0}^{K} \bar{\eta}_M v_{MN1}\right]\!\right] - \lim_{\delta \to 0} \int_{x_0-\delta}^{x_0+\delta} \bar{\rho} \sum_{M=0}^{K} \bar{\eta}_M v_{NM3}\,\mathrm{d}x = -[k_N] + \lim_{\delta \to 0} \int_{x_0-\delta}^{x_0+\delta} \bar{\rho}(s_N + \xi_N)\,\mathrm{d}x, \tag{4.6}$$

and

$$\left[\!\left[\sum_{M=0}^{K} \sum_{N=0}^{K} (\tfrac{1}{2}\bar{\rho}\bar{\boldsymbol{w}}_M \cdot \bar{\boldsymbol{w}}_N + \bar{\rho}\bar{\epsilon}_{MN})\, v_{MN1} \right]\!\right] = \left[\!\left[\sum_{N=0}^{K} (\boldsymbol{m}_N \cdot \bar{\boldsymbol{w}}_N - h_N) \right]\!\right] + \lim_{\delta\to 0} \int_{x_0-\delta}^{x_0+\delta} \sum_{N=0}^{K} (r_N + \boldsymbol{l}_N \cdot \bar{\boldsymbol{w}}_N)\, \bar{\rho}\, \mathrm{d}x, \quad (4.7)$$

where in the above formulae

$$[f] = f|_{x=x_0^+} - f|_{x=x_0^-}. \quad (4.8)$$

Before further consideration of the jump conditions in the context of particular applications in §§5–8, it is useful to examine the nature of the differential equations of motion and the jump conditions in the special case of a single constrained director for applications to problems of fluid flow over a small constant depth. As noted earlier in this section, previously both the field equations and the jump conditions were derived in lagrangian form, with the derivation of the latter being limited to that for steady flows confined to planes perpendicular to the x_2-direction. The field equations were also obtained from the eulerian formulation of the conservation laws in Green & Naghdi (1986) and it was noted that these were equivalent to the field equations in the lagrangian form.

We now proceed to obtain the field equations in the case of a single constrained director (mentioned in the preceeding paragraph) by specialization of the general results of §2 and then deduce also the corresponding jump conditions from the present eulerian forms. Thus, with reference to a fixed system of rectangular cartesian coordinate axes introduced in §3, consider a fluid sheet that is bounded below by the plane $x_3 = 0$ and above by a surface $x_3 = \phi(x_1, x_2, t)$ which represents the waveheight. The fluid is of constant mass density ρ^* and, in the absence of surface tension, flows under the constant gravitational acceleration $-g\boldsymbol{e}_3$. Also, we denote the pressure at the bottom surface by $\bar{p} = \bar{p}(x_1, x_2, t)$ and at the top surface by $\hat{p} = \hat{p}(x_1, x_2, t)$. From the general theory of §2, we select the special case of a single director and choose the weighting function $\lambda_1(\zeta) = \zeta = x_3$. Then, the velocity vector $\boldsymbol{v}$ and the director velocity $\boldsymbol{w}_1 (= \boldsymbol{w})$ when referred to the base vector $\boldsymbol{e}_i$ can be expressed as

$$\boldsymbol{v} = v_\alpha \boldsymbol{e}_\alpha, \quad \boldsymbol{w} = w_i \boldsymbol{e}_i. \quad (4.9)$$

Because the fluid is now assumed to be incompressible, from (3.20) we have

$$v_{\alpha,\alpha} + w = 0, \quad (4.10)$$

where we have also set $w_3 = w$. In addition, because the director is constrained, we have

$$w_\alpha = 0. \quad (4.11)$$

As explained in Green & Naghdi (1986, §8), the constraint conditions (4.10) and (4.11) give rise to constraint responses of the forms

$$\boldsymbol{M}_{0\alpha} = \boldsymbol{N}_\alpha = -p\boldsymbol{e}_\alpha, \quad \boldsymbol{k}_1 = -p\boldsymbol{e}_3 + r_\beta \boldsymbol{e}_\beta, \quad \boldsymbol{M}_{1\alpha} = r_{\alpha\beta}\boldsymbol{e}_\beta, \quad (4.12)$$

where $p, r_\alpha, r_{\alpha\beta}$ are arbitrary functions of x_α, t. Values for the inertia coefficients $\bar{y}_{00}, \bar{y}_{01}, \bar{y}_{11}$, the velocity components $\rho\bar{v}_{00}, \rho\bar{v}_{10}, \rho\bar{v}_{11}$ and the assigned fields $f = l_0, l_1$ were also listed in (8.7) of that paper (Green & Naghdi 1986). The relevant field equations may now be deduced from (2.12 *b*) and (2.13) and are given by

$$\left.\begin{aligned} \partial\phi/\partial t + (\phi v_\alpha)_{,\alpha} &= 0, \\ \rho^*\phi\,(\partial v_\alpha/\partial t + v_\beta \partial v_\alpha/\partial x_\beta) &= -\partial p/\partial x_\alpha + \hat{p}\,\partial\phi/\partial x_\alpha, \\ \tfrac{1}{2}\rho^*\phi^2\,(\partial w/\partial t + v_\beta \partial w/\partial x_\beta + w^2) &= -\rho^* g\phi - \hat{p} + \bar{p}, \\ \tfrac{1}{3}\rho^*\phi^3\,(\partial w/\partial t + v_\beta \partial w/\partial x_\beta + w^2) &= p - \tfrac{1}{2}\rho^* g\phi^2 - \bar{p}\phi. \end{aligned}\right\} \tag{4.13}$$

We do not record here the equations involving the response functions r_α and $r_{\alpha\beta}$ because they will not be needed in our subsequent developments. The system of equations (4.13) are equivalent to those utilized in a number of previous papers from a lagrangian formulation of the theory for an incompressible inviscid fluid.

Consider again an incompressible inviscid fluid but suppose that the two-dimensional flow (confined to the vertical x_1, x_3-plane) is steady. Then, it follows from (4.10) and $(4.13)_1$ that

$$v_1 = u = k/\phi, \quad w = k\phi'/\phi^2, \tag{4.14}$$

where u is the horizontal component of the velocity, k is a constant flow rate and a prime denotes differentiation with respect to $x_1 = x$. Now suppose that there is a discontinuity in ϕ or ϕ' at the place $x_1 = x = x_0$ (say). Then, recalling the notation (4.8), from (4.3) to (4.7) we have the jump conditions

$$[k] = 0, \tag{4.15}$$

$$[\rho^* ku] = -[p] + F_1, \tag{4.16}$$

$$[\tfrac{1}{2}\rho^* k\phi w] = F_3, \tag{4.17}$$

$$[\tfrac{1}{3}\rho^* k\phi^2 w] - \tfrac{1}{3}\lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta} \rho^*\phi^3 w^2\,\mathrm{d}x = L_3, \tag{4.18}$$

$$[\tfrac{1}{2}\rho^* k\{u^2 + \tfrac{1}{3}\phi^2 w^2 + g\phi\}] = [-pu] - \Phi, \tag{4.19}$$

where

$$\left.\begin{aligned} F_1 = \lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta} \hat{p}\phi'\,\mathrm{d}x, \quad F_3 = -\lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta} (\hat{p}-\bar{p})\,\mathrm{d}x, \\ L_3 = -\lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta} \bar{p}\phi\,\mathrm{d}x, \end{aligned}\right\} \tag{4.20}$$

and Φ is the rate of energy dissipation which may arise, for example, from spray formation at the leading edge of a boat or from local viscous effects at the jump.

It is of interest to provide a direct comparison between the jump conditions (4.15)–(4.19) and the corresponding jump conditions obtained in a number of previous papers (mentioned earlier in this section) from the lagrangian formulation of the theory. For example, apart from a slight change in notation, the jump conditions utilized in the papers of Naghdi & Rubin (1981, 1986) in a similar context can be written as

$$\left.\begin{aligned}&[k] = 0, \quad [\rho^* ku + p] = F_1, \quad [\tfrac{1}{2}\rho^* k\phi w] = F_3,\\ &[\tfrac{1}{2}\rho^* k\phi w] = L_3 - \tfrac{1}{2}\lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta} (\hat{p} + \bar{p})\,\mathrm{d}x,\\ &[\tfrac{1}{2}\rho^* k\{u^2 + (\phi w)^2 + g\phi\} + pu] = -\Phi,\end{aligned}\right\} \tag{4.21}$$

and it should be noted that w in the papers cited earlier (e.g. Naghdi & Rubin 1986) is replaced by ϕw due to a difference in notation from that used here.

The jump conditions (4.15), (4.16), (4.17) and (4.19) are the same as the corresponding conditions in (4.21), but the condition (4.18) is different. This is because the physical interpretation of the quantity L_3 in (4.18) is not the same at that of L_3 (or a combination of L_3 and F_3) in (4.21). If we examine the manner in which the inertia coefficients and the assigned fields f and l_1 have been identified (or interpreted) in both the lagrangian and eulerian formulations, then due to the use of different coordinates and weighting functions we should, in general, expect different values for those quantities to emerge. In the special case in which ϕ is continuous at $x = x_0$, but not ϕ', then (4.18) reduces to

$$\tfrac{1}{3}\rho^* k^2[\phi'] = L \tag{4.22}$$

and this is equivalent to a combination of equations $(4.21)_{2,3}$. The same is true if $\hat{p}$ is bounded in the interval $(x_0 - \delta,\, x_0 + \delta)$. In other applications it may happen that the equations yielding either the value L or L_3 are of no interest so that differences in the two formulations are of no consequences.

5. Fluids of infinite depth

Here we consider an inviscid incompressible fluid of constant density ρ^* and infinite depth using the theory of §2 and the special reference frame of §3. The fluid is subject to constant gravity $-g\boldsymbol{e}_3$ and is bounded above by the surface

$$\zeta = x_3 = \beta(x_1, x_2, t) \tag{5.1}$$

at which there is a pressure $\hat{p} = \hat{p}(x_1, x_2, t)$ and a constant surface tension T. The fluid occupies the region $-h < \zeta < \beta$ with $h \to \infty$. From §2 we select the theory with two directors corresponding to weighting functions $\lambda_1(\zeta) = \mathrm{e}^{a\zeta}$, $\lambda_2(\zeta) = \zeta$, where a is a parameter. In addition

$$\bar{\boldsymbol{d}}_0 = x_\alpha \boldsymbol{e}_\alpha + d\boldsymbol{e}_3, \quad \bar{\boldsymbol{d}}_1 = \boldsymbol{0}, \quad \bar{\boldsymbol{d}}_2 = \boldsymbol{e}_3, \tag{5.2}$$

where the scalar d is a constant. The velocity $\boldsymbol{v}$ is constrained always to be constant in a direction parallel to the plane $\bar{\sigma}$, the velocity $\boldsymbol{w}_2$ is constrained to be zero, and the fluid is incompressible so that

$$\left.\begin{aligned}\boldsymbol{v} = c_\alpha \boldsymbol{e}_\alpha, \quad \boldsymbol{w}_2 = \boldsymbol{0}, \quad \boldsymbol{w}_1 = w_i \boldsymbol{e}_i,\\ w_{\alpha,\alpha} + a w_3 = 0,\end{aligned}\right\} \tag{5.3}$$

where c_α are constants. In view of (3.21) and the extra constraints in (5.2) other than incompressibility, the constraint response functions are

$$\left.\begin{aligned}\boldsymbol{N}_\alpha = r_{i\alpha}\boldsymbol{e}_i, \quad \boldsymbol{k}_1 = -pa\boldsymbol{e}_3, \quad \boldsymbol{M}_{1\alpha} = -p\boldsymbol{e}_\alpha,\\ \boldsymbol{M}_{2\alpha} = \bar{r}_{i\alpha}\boldsymbol{e}_i, \quad \boldsymbol{k}_2 = r_i\boldsymbol{e}_i,\end{aligned}\right\} \tag{5.4}$$

where $p, r_{i\alpha}, \bar{r}_{i\alpha}, r_i$ are arbitrary functions of x_1, x_2, t. The response functions (5.2) satisfy the moment equation (2.14) if

$$r_{3\alpha} = r_\alpha, \quad r_{\alpha\beta} = r_{\beta\alpha}. \tag{5.5}$$

The mechanical field equations (2.12) and (2.13) given previously in Green & Naghdi (1986) are

$$\frac{\partial\beta}{\partial t} + (c_\alpha + w_\alpha\, e^{\alpha\beta})\frac{\partial\beta}{\partial x_\alpha} - w_3\, e^{\alpha\beta} = 0, \tag{5.6}$$

$$\frac{\rho^*}{a}\, e^{\alpha\beta}\frac{\partial w_\alpha}{\partial t} + \frac{\rho^*}{a}(e^{\alpha\beta}c_\lambda + \tfrac{1}{2}e^{2\alpha\beta}w_\lambda)\frac{\partial w_\alpha}{\partial x_\lambda} + \tfrac{1}{2}\rho^*\, e^{2\alpha\beta}w_3\, w_\alpha = \frac{\partial r_{\alpha\lambda}}{\partial x_\lambda} + (\hat{p} - q)\frac{\partial\beta}{\partial x_\alpha}, \tag{5.7}$$

$$\frac{\rho^*}{2a}\, e^{2\alpha\beta}\frac{\partial w_\alpha}{\partial t} + \frac{\rho^*}{a}(\tfrac{1}{2}e^{2\alpha\beta}c_\lambda + \tfrac{1}{3}e^{3\alpha\beta}w_\lambda)\frac{\partial w_\alpha}{\partial x_\lambda} + \tfrac{1}{3}\rho^*\, e^{3\alpha\beta}w_3\, w_\alpha = -\frac{\partial p}{\partial x_\alpha} + (\hat{p} - q)\, e^{\alpha\beta}\frac{\partial\beta}{\partial x_\alpha}, \tag{5.8}$$

$$\frac{\rho^*}{2a}\, e^{2\alpha\beta}\frac{\partial w_3}{\partial t} + \frac{\rho^*}{a}(\tfrac{1}{2}e^{2\alpha\beta}c_\lambda + \tfrac{1}{3}e^{3\alpha\beta}w_\lambda)\frac{\partial w_3}{\partial x_\lambda} + \tfrac{1}{3}\rho^*\, e^{3\alpha\beta}w_3^2 = ap - (\rho^* g/a)\, e^{\alpha\beta} - (\hat{p} - q)\, e^{\alpha\beta}. \tag{5.9}$$

Equations involving the response functions $r_\alpha, \bar{r}_{3\alpha}$ are not required in subsequent analysis and are omitted. Also

$$\frac{q}{T} = \frac{\{1 + (\beta_{,1})^2\}\beta_{,22} - 2\beta_{,1}\beta_{,2}\beta_{,12} + \{1 + (\beta_{,2})^2\}\beta_{,11}}{\{1 + (\beta_{,1})^2 + (\beta_{,2})^2\}^{\frac{3}{2}}}. \tag{5.10}$$

We consider steady waves on a stream moving with constant speed c in the x_1-direction, with the temperature everywhere constant. Then, $c_1 = c, w_2 = 0$ and $\beta, w_1, w_3, p, r_{\lambda\alpha}$ are functions of x_1. With zero surface tension, $(5.3)_4$ and (5.6) to (5.9) reduce to

$$w_1' + aw_3 = 0, \quad (c + w_1\, e^{\alpha\beta})\beta' - w_3\, e^{\alpha\beta} = 0, \tag{5.11}$$

$$\rho^* c\, e^{2\alpha\beta}w_1'/(2a) = -p' + \hat{p}\, e^{\alpha\beta}\beta', \tag{5.12}$$

$$\frac{\rho^* c}{2a}\, e^{2\alpha\beta}w_3' + \frac{\rho^*\, e^{3\alpha\beta}}{3a}(w_1 w_3' + aw_3^2) = ap - (\rho^* g/a)\, e^{\alpha\beta} - \hat{p}\, e^{\alpha\beta}, \tag{5.13}$$

$$\rho^* c\, e^{\alpha\beta}w_1'/a = r_{11}' + \hat{p}\beta', \tag{5.14}$$

where a prime denotes differentiation with respect to x_1. Equations (5.11) may be integrated to yield

$$w_1 = c(A - a\beta)\, e^{-\alpha\beta}, \quad w_3 = c(1 + A - a\beta)\, e^{-\alpha\beta}\beta', \tag{5.15}$$

where A is a constant.

Suppose that there is a discontinuity in β or β' or both at the place $x = x_0$ which induces discontinuities in the dynamical variables. Then, from (4.3) to (4.8) and the equations of the present section we have the following jump conditions

$$[ca\beta + w_1\, e^{\alpha\beta}] = [cA] = 0, \tag{5.16}$$

$$[(\rho^*/a)\,(c\, e^{\alpha\beta} + \tfrac{1}{2}w_1\, e^{2\alpha\beta})\, w_1] = [r_{11}] + F_1, \tag{5.17}$$

$$[(\rho^*/a)\,(c\, e^{\alpha\beta} + \tfrac{1}{2}w_1\, e^{2\alpha\beta})c + (\rho^*/a)\,(\tfrac{1}{2}c\, e^{2\alpha\beta} + \tfrac{1}{3}w_1\, e^{3\alpha\beta})\, w_1] - \lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta}\rho^*(\tfrac{1}{2}c\, e^{2\alpha\beta} + \tfrac{1}{3}w_1\, e^{3\alpha\beta})\, w_3\, dx = -[p] + L_1, \tag{5.18}$$

$$[(\rho^*/a)\,(\tfrac{1}{2}c\, e^{2\alpha\beta} + \tfrac{1}{3}w_1\, e^{3\alpha\beta})\, w_3] - \lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta}\tfrac{1}{3}\rho^*\, e^{3\alpha\beta}w_3^2\, dx = L_3 + \lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta}\{ap - (\rho^* g/a)\, e^{\alpha\beta}\}\, dx, \tag{5.19}$$

$$[(\rho^*/a)\,(c\,\mathrm{e}^{a\beta}+\tfrac{1}{2}w_1\,\mathrm{e}^{2a\beta})\,cw_1+(\rho^*/2a)\,(\tfrac{1}{2}c\,\mathrm{e}^{2a\beta}+\tfrac{1}{3}w_1\,\mathrm{e}^{3a\beta})\,(w_1^2+w_3^2)]$$
$$=[cr_{11}-pw_1]-c\Phi-\lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta}(\rho^*/a)\,g\,\mathrm{e}^{a\beta}w_3\,\mathrm{d}x, \quad (5.20)$$

where $c\Phi$ is a dissipation of energy which may arise, for example, from spray formation at the leading edge of the boat or from local viscous effects at the jump and

$$F_1=\lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta}\hat{p}\beta'\,\mathrm{d}x,\quad L_1=\lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta}\hat{p}\,\mathrm{e}^{a\beta}\beta'\,\mathrm{d}x,\quad L_3=-\lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta}\hat{p}\,\mathrm{e}^{a\beta}\,\mathrm{d}x. \quad (5.21)$$

6. Wedge-like boat

Problems of steady two-dimensional planing on water of large depth have been considered by Wagner (1932), Squire (1957) and Cumberbatch (1958) using the linearized three-dimensional equations of an incompressible inviscid fluid and small planing angles. On the other hand the nonlinear steady-state solution to the problem of gliding of a wedge like boat or plate on a fluid of large depth has been given by Green & Naghdi (1986) based on equations of the form given in §5 of this paper. Here we complete the discussion of this problem by examining the transition to planing of a wedge-like boat, using the theory of §5 in which use is also made of the jump conditions. The method of solution is similar to that used by Naghdi & Rubin (1981, 1986) for the transition to planing on fluids of small depth, and may be extended in the case of water of infinite depth to apply to boats of a general shape.

We consider the equivalent problem of steady motion of a fluid in the $x=x_1$-direction past a wedge-shaped boat (see figure 1). Far ahead of the boat the fluid is assumed to flow as a

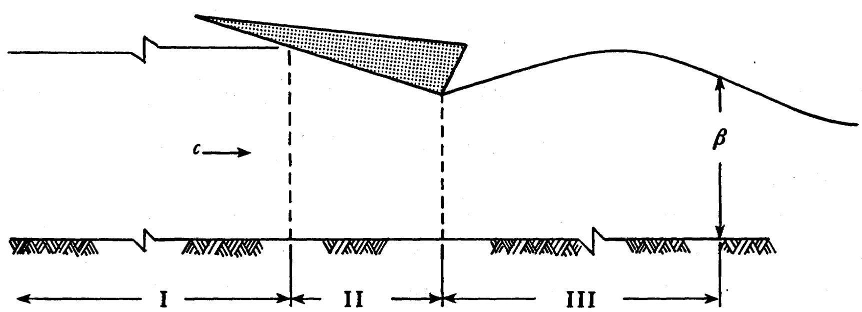

Figure 1. A sketch of the planing of a wedge-like boat showing different regions referred to in the text (see §6).

uniform stream with speed c until it meets the leading edge of the boat at $x=x_1$. The motion is two-dimensional in the (x_1,x_3) plane. It is convenient to divide the fluid into three regions: region I: $-\infty<x\leqslant x_I$; region II: $x_I\leqslant x\leqslant x_{II}$ and region III: $x_{II}\leqslant x<\infty$. The fluid height in region I is $\beta=0$. In region II the wetted length of the wedge is l and the fluid height at the trailing edge $x=x_{II}$ is $\beta_2=-l\sin\alpha$ where α is the inclination of the boat bottom to the horizontal. The fluid is assumed to leave the trailing edge of the boat smoothly and forms a wavey system in region III. The pressure at the fluid surface in regions I, III is constant and equal to p_0 and $\hat{p}$ is the varying surface pressure at the boat in region *II* which has to be found. Under these conditions the equations of motion (5.12) to (5.14) may be integrated to yield:

region *I*, $$\beta = 0, \quad w_1 = w_3 = 0, \quad p = p_0/a + \rho^* g/a^2; \tag{6.1}$$

region *II*, $$w_1 = c(A_2 - a\beta)\,\mathrm{e}^{-a\beta}, \quad w_3 = c(1 + A_2 - a\beta)\,\mathrm{e}^{-a\beta}\beta', \tag{6.2a}$$

$$\begin{aligned} p/\rho^* = {} & (c^2/2a)\,\{2 + A_2 + (1 + A_2)\,a\beta - \tfrac{1}{2}a^2\beta^2\}\,\mathrm{e}^{a\beta} - (g/a^2)\,a\beta\,\mathrm{e}^{a\beta} \\ & + (c^2/6a)\,\{(4 + 3A_2)\,a\beta - 3a^2\beta^2/2\}\,\mathrm{e}^{a\beta}\tan^2\alpha + E_2\,\mathrm{e}^{a\beta}, \end{aligned} \tag{6.2b}$$

$$\begin{aligned} \hat{p}/\rho^* = {} & (c^2/2)\,\{2 + A_2 + (1 + A_2)\,a\beta - \tfrac{1}{2}a^2\beta^2\} - (g/a)\,(1 + a\beta) \\ & + (c^2/6)\,\{4 + 3A_2 + (1 + 3A_2)\,a\beta - 3a^2\beta^2/2\}\tan^2\alpha + aE_2; \end{aligned} \tag{6.2c}$$

region *III*,

$$w_1 = c(A_3 - a\beta)\,\mathrm{e}^{-a\beta}, \quad w_3 = c(1 + A_3 - a\beta)\,\mathrm{e}^{-a\beta}\beta', \tag{6.3a}$$

$$\begin{aligned} \tfrac{1}{12}\Gamma(\beta')^2(1 + A_3 - a\beta')^2(3 + 2A_3 - 2a\beta) = {} & \tfrac{1}{2}\Gamma\{(1 + A_3)\,(2 + A_3)\,a\beta \\ & - \tfrac{1}{2}(3 + 2A_3)\,a^2\beta^2 + \tfrac{1}{3}a^3\beta^3\} - (1 + A_3)\,a\beta + \tfrac{1}{2}a^2\beta^2 - D_3(A_3 - a\beta)\,\mathrm{e}^{-a\beta} + E_3, \end{aligned} \tag{6.3b}$$

$$p/\rho^* = (p_0/\rho^* a)\,\mathrm{e}^{a\beta} + (c^2/2a)\,(2 + A_3 - a\beta)\,\mathrm{e}^{a\beta} + gD_3/a^2, \tag{6.3c}$$

where $\Gamma = c^2a/g$ and A_2, A_3, D_3, E_2, E_3 are constants.

At the leading edge $x = x_I$ between regions *I*, *II* there is a jump in the values of β' whereas β is continuous. We apply the jump conditions (5.16) to (5.20) so that

$$\left.\begin{aligned} & A_2 = 0, \quad [r_{11}] + F_1 = 0, \quad [p] = L_1, \\ & L_3 = -\frac{\rho^* c^2}{2a}\tan\alpha, \quad [cr_{11}] - c\Phi = \frac{\rho^* c^3}{4a}\tan^2\alpha, \quad L_1 = F_1. \end{aligned}\right\} \tag{6.4}$$

Hence $$E_2 = -\frac{\Phi}{\rho^*} + \frac{p_0}{\rho^* a} + \frac{g}{a^2}(1 - \Gamma - \tfrac{1}{4}\Gamma\tan^2\alpha). \tag{6.5}$$

At this edge the plate is acted on by an isolated force with horizontal and vertical components $\mathscr{D}_R, \mathscr{L}_R$ respectively, where

$$\mathscr{D}_R = -F_1 = \frac{\rho^* c^2}{4a}\tan^2\alpha + \Phi, \quad \mathscr{L}_R = -L_3 = \frac{\rho^* c^2}{2a}\tan\alpha. \tag{6.6}$$

At the trailing edge $x = x_{II}, \beta = -b\tan\alpha$ where $b = l\cos\alpha$, we assume that the flow is smooth with $\beta' = -\tan\alpha$, and conditions (5.16) to (5.20) give

$$A_3 = 0, \quad [p] = 0, \tag{6.7}$$

so that, with the help of (6.2*b*), (6.3*b*) and (6.3*c*) we have

$$\begin{aligned} & D_3\,\mathrm{e}^{ab\tan\alpha} = \frac{a^2}{g}\left(E_2 - \frac{p_0}{\rho^* a}\right) + ab(1 - \Gamma)\tan\alpha - \Gamma(\tfrac{1}{4}a^2b^2\tan^2\alpha + \tfrac{2}{3}ab\tan^3\alpha + \tfrac{1}{4}a^2b^2\tan^4\alpha), \\ & E_3 = (D_3\,ab\tan\alpha)\,\mathrm{e}^{ab\tan\alpha} - ab\tan\alpha - \tfrac{1}{2}a^2b^2\tan^2\alpha + \Gamma(ab\tan\alpha \\ & \qquad + \tfrac{3}{4}a^2b^2\tan^2\alpha + \tfrac{1}{6}a^3b^3\tan^3\alpha) + \frac{\Gamma}{12}(1 + ab\tan\alpha)^2(3 + 2ab\tan\alpha)\tan^2\alpha. \end{aligned} \tag{6.8}$$

By (6.5), (6.8) reduces to

$$D_3\,\mathrm{e}^{ab\tan\alpha} = -\frac{a^2\Phi}{\rho^* g}+1-\Gamma+(1-\Gamma)\,ab\tan a$$
$$-\Gamma\{\tfrac{1}{4}(1+a^2b^2)\tan^2\alpha+\tfrac{2}{3}ab\tan^3\alpha+\tfrac{1}{4}a^2b^2\tan^4\alpha\}$$
$$E_3 = -\frac{\Phi a^3 b\tan\alpha}{\rho^* g}+\tfrac{1}{4}(2-\Gamma)\,a^2b^2\tan^2\alpha-\frac{\Gamma ab}{12}(3+a^2b^2)\tan^3\alpha-\frac{2\Gamma a^2b^2}{3}\tan^4\alpha$$
$$-\tfrac{1}{4}\Gamma a^3b^3\tan^5\alpha+\frac{\Gamma}{12}(1+ab\tan\alpha)^2(3+2ab\tan\alpha)\tan^2\alpha. \quad (6.9)$$

With the help of (6.5) the pressure at the bottom of the boat, given by (6.2*c*), becomes

$$\frac{(\hat{p}-p_0)\,a}{\rho^* g} = -\frac{a^2\Phi}{\rho^* g}-\tfrac{1}{2}(2-\Gamma)\,a\beta-\tfrac{1}{4}\Gamma a^2\beta^2+\frac{\Gamma}{12}(5+2a\beta-3a^2\beta^2)\tan^2\alpha. \quad (6.10)$$

The total force, per unit width, normal to the boat bottom due to this pressure difference is

$$L = \int_{-b\tan\alpha}^{0}\frac{(\hat{p}-p_0)\,\mathrm{d}\beta}{\sin\alpha}$$
$$= -ab\Phi\sec\alpha+\frac{\rho^* c^2}{a\Gamma\sin\alpha}\Big\{\frac{2-\Gamma}{4}a^2b^2\tan^2\alpha$$
$$+\frac{\Gamma}{12}[ab(5-a^2b^2)-a^2b^2\tan\alpha-a^3b^3\tan^3\alpha]\tan^3\alpha\Big\}. \quad (6.11)$$

Wave motion in region *III* is governed by (6.3*b*) in which $A_3 = 0$ and D_3, E_3 are given by (6.9). Such motion will only be possible if $\Gamma < 2$. To complete the problem it is necessary to specify the values of Γ (or of a) and Φ. Guided by the pressure problem studied by Green & Naghdi (1986) we choose

$$\Gamma = 1, \quad a = g/c^2. \quad (6.12)$$

If a Froude number F for the flow is defined in terms of the wetted length of the boat, i.e. $F^2 = c^2/(gl)$ then, from (6.12), the normal force (6.11), and the components of the isolated force (6.6), become

$$\left.\begin{aligned} L &= -\frac{\Phi}{F^2}+\frac{\rho^* g l^2\tan\alpha}{4}\Big\{\cos\alpha+\frac{1}{3}\Big[5F^2-\frac{\cos^2\alpha}{F^2}-\sin\alpha-\frac{\sin^3\alpha}{F^2\cos\alpha}\Big]\tan\alpha\Big\},\\ \mathscr{D}_R &= \tfrac{1}{4}\rho^* g l^2F^4\tan^2\alpha+\Phi, \quad \mathscr{L}_R = \tfrac{1}{2}\rho^* g l^2F^4\tan\alpha. \end{aligned}\right\} \quad (6.13)$$

Following the discussion of Naghdi & Rubin (1986) for the boat problem on water of small depth, we choose Φ so that the resultant of all the forces in (6.13) is normal to the plate. Then

$$\Phi = \tfrac{1}{4}\rho^* g l^2F^4\tan^2\alpha, \quad \mathscr{D}_R = \tfrac{1}{2}\rho^* g l^2F^4\tan^2\alpha \quad (6.14)$$

and the total resultant force normal to the plate is

$$L_T = L+\mathscr{D}_R\sin\alpha+\mathscr{L}_R\cos\alpha$$
$$= \frac{\rho^* g l^2\tan\alpha}{4}\Big\{\cos\alpha+2F^4\sec\alpha+\frac{1}{3}\Big[2F^2-\frac{\cos^2\alpha}{F^2}-\sin\alpha-\frac{\sin^3\alpha}{F^2\cos\alpha}\Big]\tan\alpha\Big\}. \quad (6.15)$$

In a similar way the moment of the fluid forces on the plate, about any suitable point, may be computed. Then a discussion of the motion of the boat under suitable external forces may be carried out in a manner similar to that given by Naghdi & Rubin (1986) for water of small depth, but we omit details.

If the angle of inclination α of the bottom plane of the boat is small then the wave elevation in region *III*, given by (6.3*b*), (6.9) and (6.14), can be expressed as a series of $\tan\alpha$, the major term being

$$\beta = -\{a^{-1}\sin a(x-x_{II}) + l\cos\alpha\cos a(x-x_{II})\}\tan\alpha, \tag{6.16}$$

where $a^{-1} = F^2 l$.

7. Fluids of finite depth

We continue to use the cartesian coordinate system introduced in §3 and we consider a fluid of constant density ρ^* bounded below by the fixed plane $x_3 = -h$, where h is a constant, and with the surface of the stream given by

$$\zeta = x_3 = \beta(x_1, x_2, t)$$

at which there is a pressure $\hat{p} \equiv \hat{p}(x_1, x_2, t)$ and a constant surface tension T. The pressure at the bottom surface is $\bar{p}(x_1, x_2, t)$. From §2 we select the theory with three directors corresponding to weighting functions $\lambda_1(\zeta) = \cosh a(\zeta+h)$, $\lambda_2(\zeta) = \sinh a(\zeta+h)$, $\lambda_3(\zeta) = \zeta$, with a being constant and we choose

$$\bar{\boldsymbol{d}}_0 = x_\alpha \boldsymbol{e}_\alpha + d\boldsymbol{e}_3, \quad \bar{\boldsymbol{d}}_1 = \bar{\boldsymbol{d}}_2 = 0, \quad \bar{\boldsymbol{d}}_3 = \boldsymbol{e}_3, \tag{7.1}$$

where d is a constant. The velocity $\boldsymbol{v}$ is constrained always to be constant in a direction parallel to the plane $\bar{\sigma}$, the velocity $\bar{\boldsymbol{w}}_3$ is constrained to be zero, and the fluid is incompressible so that

$$\left.\begin{aligned} &\boldsymbol{v} = c_\alpha \boldsymbol{e}_\alpha, \quad \bar{\boldsymbol{w}}_3 = 0, \quad \boldsymbol{w}_1 = w_{1i}\boldsymbol{e}_i, \quad \boldsymbol{w}_2 = w_{2i}\boldsymbol{e}_i, \\ &w_{1\alpha,\alpha} + a w_{23} = 0, \quad w_{2\alpha,\alpha} + a w_{13} = 0, \end{aligned}\right\} \tag{7.2}$$

where c_α are constants. We also impose the additional constraint

$$w_{2\alpha} = 0 \tag{7.3}$$

on the second director. In view of (3.21) and the extra constraints other than incompressibility, the response functions are

$$\left.\begin{aligned} &\boldsymbol{N}_\alpha = r_{i\alpha}\boldsymbol{e}_i, \quad \boldsymbol{k}_1 = -p_1 a\boldsymbol{e}_3, \quad \boldsymbol{k}_2 = q_\alpha \boldsymbol{e}_\alpha - pa\boldsymbol{e}_3, \\ &\boldsymbol{M}_{1\alpha} = -p\boldsymbol{e}_\alpha, \quad \boldsymbol{M}_{2\alpha} = -q_{\beta\alpha}\boldsymbol{e}_\beta - p_1\boldsymbol{e}_\alpha, \quad \boldsymbol{M}_{3\alpha} = s_{i\alpha}\boldsymbol{e}_i, \quad \boldsymbol{k}_3 = s_i\boldsymbol{e}_i, \end{aligned}\right\}, \tag{7.4}$$

where $r_{i\alpha}, q_\alpha, q_{\alpha\beta}, p, p_1, s_{i\alpha}, s_i$ are functions of x_α, t. In view of (7.1) the response functions (7.4) satisfy the moment equations (2.14) if

$$r_{3\alpha} = s_\alpha, \quad r_{\alpha\beta} = r_{\beta\alpha}. \tag{7.5}$$

The mechanical field equations (2.12) and (2.13) given previously in Green & Naghdi (1986) are

$$\frac{\partial\phi}{\partial t} + \{c_\alpha + w_{1\alpha}\cosh(a\phi)\}\frac{\partial\phi}{\partial x_\alpha} - w_{23}\sinh(a\phi) = 0, \tag{7.6}$$

$$\begin{aligned} \frac{\rho^*}{a}\sinh(a\phi)\frac{\partial w_{1\alpha}}{\partial t} + \frac{\rho^*}{a}\{c_\beta\sinh(a\phi) + \tfrac{1}{2}[\sinh(a\phi)\cosh(a\phi) + a\phi]w_{1\beta}\}\frac{\partial w_{1\alpha}}{\partial x_\beta} \\ + \tfrac{1}{2}\rho^*\{\sinh(a\phi)\cosh(a\phi) - a\phi\}w_{23}w_{1\alpha} = \frac{\partial r_{\alpha\beta}}{\partial x_\beta} + (\hat{p} - q)\frac{\partial\phi}{\partial x_\alpha}, \end{aligned} \tag{7.7}$$

$$\frac{\rho^*}{2a}\{\sinh(a\phi)\cosh(a\phi)+a\phi\}\frac{\partial w_{1\alpha}}{\partial t}+\frac{\rho^*}{a}\{\tfrac{1}{2}c_\beta[\sinh(a\phi)\cosh(a\phi)+a\phi]+[\tfrac{1}{3}\sinh^3(a\phi)$$
$$+\sinh(a\phi)]\,w_{1\beta}\}\frac{\partial w_{1\alpha}}{\partial x_\beta}+\tfrac{1}{3}\rho^*\sinh^3(a\phi)\,w_{23}w_{1\alpha}=-\frac{\partial p}{\partial x_\alpha}+(\hat{p}-q)\cosh(a\phi)\frac{\partial\phi}{\partial x_\alpha}, \quad (7.8)$$

$$\frac{\rho^*}{2a}\{\sinh(a\phi)\cosh(a\phi)-a\phi\}\frac{\partial w_{23}}{\partial t}$$
$$+\frac{\rho^*}{a}\{\tfrac{1}{2}c_\alpha[\sinh(a\phi)\cosh(a\phi)-a\phi]+\tfrac{1}{3}\sinh^3(a\phi)\,w_{1\alpha}\}\frac{\partial w_{23}}{\partial x_\alpha}$$
$$+\frac{\rho^*}{3}\sinh^3(a\phi)\,w_{23}^2=ap-(\hat{p}-q)\sinh(a\phi)-\rho^*(g/a)\{\cosh(a\phi)-1\}. \quad (7.9)$$

Equations involving the constraint response functions $p_1, r_{3\alpha}, q_\alpha, q_{\alpha\beta}, s_{t\alpha}, s_t$ are omitted because the values of these constraint response functions are not required in the further development of the theory. Also $\phi=\beta+h$ and q is given by (5.10).

We consider steady motion on a stream moving with constant speed c in the x_1-direction with the temperature everywhere constant and zero surface tension. Equations (7.2) and (7.6) to (7.9) reduce to

$$\left.\begin{aligned}w_{11}&=c(A-a\beta)/\sinh(a\phi),\\ w_{23}&=c\{(A-a\beta)\cosh(a\phi)+\sinh(a\phi)\}\phi'/\sinh^2(a\phi),\end{aligned}\right\} \quad (7.10)$$

$$(\rho^*/a)\,(c\sinh(a\phi)+a\phi w_{11})\,w_{11}'=r_{11}'+\hat{p}\phi', \quad (7.11)$$

$$(\rho^*/2a)\,\{c[(\sinh(a\phi)\cosh(a\phi)+a\phi]+2w_{11}\sinh(a\phi)\}\,w_{11}'=-p'+\hat{p}\cosh(a\phi)\,\phi', \quad (7.12)$$

$$(\rho^*/a)\,\{\tfrac{1}{2}c[\sinh(a\phi)\cosh(a\phi)-a\phi]+\tfrac{1}{3}\sinh^3(a\phi)w_{11}\}\,w_{23}'$$
$$+\tfrac{1}{3}\rho^*\sinh^3(a\phi)\,w_{23}^2=ap-\hat{p}\sinh(a\phi)-\rho^*(g/a)\{\cosh(a\phi)-1\}, \quad (7.13)$$

where A is a constant and a prime denotes differentiation with respect to x_1.

Suppose that there is a discontinuity in β or β' or both at the place $x=x_0$ which induces discontinuities in the dynamical variables. Then, from (4.3) to (4.8) and the equations of the present section we have the following jump conditions:

$$[ca\phi+w_{11}\sinh(a\phi)]=[c(A+ah)]=[cA]=0, \quad (7.14)$$

$$[(\rho^*/a)\{c\sinh(a\phi)+\tfrac{1}{2}[\sinh(a\phi)\cosh(a\phi)+a\phi]\,w_{11}\}\,w_{11}]=[r_{11}]+F_1, \quad (7.15)$$

$$[(\rho^*/a)\{c\sinh(a\phi)+\tfrac{1}{2}[\sinh(a\phi)\cosh(a\phi)+a\phi]\,w_{11}\}\,c$$
$$+(\rho^*/a)\,\{\tfrac{1}{2}c[\sinh(a\phi)\cosh(a\phi)+a\phi]+[\tfrac{1}{3}\sinh^3(a\phi)+\sinh(a\phi)]\,w_{11}\}\,w_{11}]$$
$$-\lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta}\{\tfrac{1}{2}\rho^*[\sinh(a\phi)\cosh a\phi-a\phi]\,c+\tfrac{1}{3}\rho^*\sinh^3(a\phi)\,w_{11}\}\,w_{23}\,dx=-[p]+\bar{L}_1, \quad (7.16)$$

$$[(\rho^*/a)\{\tfrac{1}{2}c[\sinh(a\phi)\cosh(a\phi)-a\phi]+\tfrac{1}{3}\sinh^3(a\phi)w_{11}\}\,w_{23}]$$
$$-\lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta}\tfrac{1}{3}\rho^*\sinh^3(a\phi)\,w_{23}^2\,dx=\bar{L}_3-\lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta}\{ap+(\rho^*g/a)\,[\cosh(a\phi)-1]\}\,dx, \quad (7.17)$$

$$[(\rho^*/a)\{c\sinh(a\phi)+\tfrac{1}{2}[\sinh(a\phi)\cosh(a\phi)+a\phi]\,w_{11}\}\,cw_{11}$$
$$+(\rho^*/2a)\{\tfrac{1}{2}c[\sinh(a\phi)\cosh(a\phi)+a\phi]+[\tfrac{1}{3}\sinh^3(a\phi)+\sinh(a\phi)]\,w_{11}\}\,w_{11}^2$$
$$+(\rho^*/2a)\{\tfrac{1}{2}c[\sinh(a\phi)\cosh(a\phi)-a\phi]+\tfrac{1}{3}\sinh^3(a\phi)\,w_{11}\}\,w_{23}^2]$$
$$=[cr_{11}-pw_{11}]-c\Phi-\lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta}(\rho^*/a)\,g\{\cosh(a\phi)-1\}\,w_{23}\,\mathrm{d}x, \quad (7.18)$$

where $c\Phi$ is a dissipation of energy, F_1 is given by $(5.21)_1$ and

$$\bar{L}_1=\lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta}\hat{p}\phi'\cosh(a\phi)\,\mathrm{d}x, \quad \bar{L}_3=-\lim_{\delta\to 0}\int_{x_0-\delta}^{x_0+\delta}\hat{p}\sinh(a\phi)\,\mathrm{d}x. \quad (7.19)$$

8. Wedge-like boat for finite depth

The problem of the transition to planing for boats, including wedge-like boats, has been solved by Naghdi & Rubin (1981, 1986) using a theory appropriate for a fluid of small depth. The corresponding problem for a fluid of large depth is discussed in §6. Here we consider the same problem by using the slightly more complex theory of §7 which is appropriate for fluids of any depth, including as limiting cases the two previous problems.

As in §6, we consider the steady flow of the fluid in the $x=x_1$-direction past a wedge-shaped boat as depicted in figure 1, but bounded below by a horizontal plane. We again divide the fluid into three regions as described in §6 so that in each region we have:

region *I*,

$$\beta=0,\quad \phi=h,\quad w_{11}=w_{23}=0,\quad p=(p_0/a)\sinh(ah)+(\rho^*g/a^2)[\cosh(a\phi)-1]; \quad (8.1)$$

region *II*,

$$\left.\begin{aligned}w_{11}=c(A_2-a\beta)/\sinh(a\phi),\quad w_{23}=c\{(A_2-a\beta)\cosh(a\phi)+\sinh(a\phi)\}\phi'/\sinh^2(a\phi)\\ \phi'=-\tan\alpha,\end{aligned}\right\} \quad (8.2a)$$

$$p=\frac{\rho^*gB_2}{a^2}\sinh(a\phi)-\frac{\rho^*g}{a^2}\{a\phi\sinh(a\phi)-\cosh(a\phi)+1\}-\frac{\rho^*c^2(A_2-a\beta)^2}{2a\sinh(a\phi)}$$
$$+\frac{\rho^*c^2}{a}\left\{(A_2-a\beta)\left[-\frac{a\phi}{4\sinh(a\phi)}+\frac{a\phi\sinh(a\phi)}{2}-\frac{3\cosh(a\phi)}{4}\right]\right.$$
$$\left.+\frac{a^2\phi^2\sinh(a\phi)}{4}-\frac{a\phi\cosh(a\phi)}{4}\right\}+G(\phi)\sinh(a\phi)\tan^2\alpha, \quad (8.2b)$$

$$\hat{p}=\frac{\rho^*gB_2}{a}-\frac{\rho^*ga\phi}{a}-\frac{\rho^*c^2(A_2-a\beta)^2}{2\sinh^2(a\phi)}$$
$$+\rho^*c^2\left\{(A_2-a\beta)\left[-\frac{a\phi}{4\sinh^2(a\phi)}+\frac{a\phi}{2}-\frac{3\cosh(a\phi)}{4\sinh(a\phi)}\right]\right.$$
$$\left.+\frac{a^2\phi^2}{4}-\frac{a\phi\cosh(a\phi)}{4\sinh(a\phi)}\right\}+\left\{aG+\frac{\mathrm{d}G}{\mathrm{d}\phi}\tanh(a\phi)\right\}\tan^2\alpha, \quad (8.2c)$$

$$\frac{\mathrm{d}G}{\mathrm{d}\phi} = -\frac{\cosh(a\phi)}{6a^3\sinh^4(a\phi)}\{[2c\cosh(a\phi)+w_{11}(\cosh^2(a\phi)+1)]\,[c(3a\phi-3\sinh(a\phi)\cosh(a\phi)$$
$$-2w_{11}\sinh^3(a\phi)]+2\sinh^3(a\phi)\,[c+w_{11}\cosh(a\phi)]^2\}; \quad (8.2d)$$

region *III*,

$$w_{11}=c(A_3-a\beta)/\sinh(a\phi), \quad w_{23}=c\{(A_3-a\beta)\cosh(a\phi)+\sinh(a\phi)\phi'/\sinh^3(a\phi), \quad (8.3a)$$

$$\tfrac{1}{12}\Gamma(\phi')^2\left\{\frac{(A_3-a\beta)\cosh(a\phi)+\sinh(a\phi)}{\sinh^2(a\phi)}\right\}^2$$
$$\times\{3\sinh(a\phi)\cosh(a\phi)-3a\phi+2(A_3-a\beta)\sinh^2(a\phi)\}$$
$$=-\frac{(A_3-a\beta)\{B_3+1-\cosh(a\phi)\}}{\sinh(a\phi)}+\tfrac{1}{2}(A_3-a\beta)^2+\Gamma\left\{\frac{(A_3-a\beta)^2[2A_3-a(\beta-h)]}{4\sinh^2(a\phi)}\right.$$
$$\left.-\frac{3(A_3-a\beta)^2\cosh(a\phi)}{4\sinh(a\phi)}-\tfrac{1}{6}(A_3-a\beta)^3-(A_3-a\beta)\right\}+E_3, \quad (8.3b)$$

$$p=\frac{\rho^* g B_3}{a^2}-\frac{\rho^* c}{2a}\{\sinh(a\phi)\cosh(a\phi)+a\phi+2(A_3-a\beta)\}\,w_{11}$$
$$+\frac{\rho^* c^2}{a}\{(A_3-a\beta)\cosh(a\phi)+\sinh(a\phi)\}+\frac{p_0}{a}\sinh(a\phi), \quad (8.3c)$$

where $\Gamma=c^2a/g$.

At the leading edge $x=x_I$ between regions *I* and *II* there is a jump in the value of ϕ' whereas ϕ is continuous. Applying the jump conditions (7.14) to (7.19) when $\phi=h$ yields the following relations:

$$\left.\begin{aligned} &A_2=0, \quad [r_{11}]+F_1=0, \quad [p]=\bar{L}_1,\\ &\bar{L}_3=-\frac{\rho^* c^2}{2a}\left\{\frac{\sinh(ah)\cosh(ah)-ah}{\sinh(ah)}\right\}\tan\alpha, \quad \bar{L}_1=F_1\cosh(ah),\\ &[cr_{11}]-c\Phi=\frac{\rho^* c^3}{4a}\left\{\frac{\sinh(ah)\cosh(ah)-ah}{\sinh^2(ah)}\right\}\tan^2\alpha.\end{aligned}\right\} \quad (8.4)$$

Hence
$$\frac{[p]}{\cosh(ah)}=-\Phi-\frac{\rho^* c^2}{4a}\left\{\frac{\sinh(ah)\cosh(ah)-ah}{\sinh^2(ah)}\right\}\tan^2 a \quad (8.5)$$

and, with the help of (8.1) and (8.2*b*), it follows that

$$\left(B_2-\frac{p_0 a}{\rho^* g}\right)\tanh(ah)=ah\tanh(ah)-\frac{\Gamma}{4}\{a^2h^2\tanh(ah)-ah\}$$
$$-\frac{\Gamma}{4}\left\{\frac{\sinh(ah)\cosh(ah)-ah}{\sinh^2(ah)}\right\}\tan^2\alpha-\frac{a^2}{\rho^* g}F(h)\tanh(ah)\tan^2\alpha-\frac{a^2\Phi}{\rho^* g}. \quad (8.6)$$

At this leading edge, the boat is acted on by an isolated force with horizontal and vertical components $\mathscr{D}_R$, $\mathscr{L}_R$ respectively, where

$$\left.\begin{aligned} &\mathscr{D}_R=-F_1=\Phi+\frac{\rho^* c^2}{4a}\left\{\frac{\sinh(ah)\cosh(ah)-ah}{\sinh^2(ah)}\right\}\tan^2\alpha,\\ &\mathscr{L}_R=-\frac{\bar{L}_3}{\sinh(ah)}=\frac{\rho^* c^2}{2a}\left\{\frac{\sinh(ah)\cosh(ah)-ah}{\sinh^2(ah)}\right\}\tan\alpha.\end{aligned}\right\} \quad (8.7)$$

At the trailing edge $x = x_{II}$, $\beta = -b\tan\alpha = -l\sin\alpha$, we assume that the flow separates smoothly with $\beta' = \phi' = -\tan\alpha$, and conditions (7.14) to (7.19) give

$$A_3 = 0, \quad [p] = 0 \tag{8.8}$$

so that, with the help of (8.2*b*), (8.3*b*) and (8.3*c*), we have

$$B_3 = \cosh(ah) - 1 - \Gamma\sinh(ah) - \frac{a^2\Phi\coth(ah)}{\rho^* g} + O(\tan^2\alpha),$$

$$E_3 = \frac{\Phi a^3 b\cosh(ah)\tan\alpha}{\rho^* g\sinh^2(ah)}\left[1 - \frac{ab\cosh(ah)\tan\alpha}{\sinh(ah)}\right] + a^2b^2\left[\frac{1}{2} - \frac{\Gamma}{4}\left\{\frac{ah + \sinh(ah)\cosh(ah)}{\sinh^2(ah)}\right\}\right]\tan^2\alpha$$
$$+\frac{\Gamma[\sinh(ah)\cosh(ah) - ah]\tan^2\alpha}{4\sinh^2(ah)} + O(\tan^3\alpha). \tag{8.9}$$

The constants have only been calculated explicitly to the order stated since, in the rest of the calculations we shall assume that the angle α is small.

With the help of (8.6) and (8.2*c*) the total force per unit width, normal to the boat bottom due to the pressure difference $\hat{p} - p_0$ is

$$L = \int_{-b\tan\alpha}^{0} \frac{(\hat{p} - p)\,d\beta}{\sin\alpha} = \frac{-\Phi\Gamma\cosh(ah)}{F^2\sinh(ah)}$$
$$+\frac{\rho^* g l^2}{2}\left[1 - \frac{\Gamma\{\sinh(ah)\cosh(ah) + ah\}}{2\sinh^2(ah)}\right]\sin\alpha + O(\tan^2\alpha), \tag{8.10}$$

where $F^2 = c^2/(gl)$.

Wave motion in region *III* is governed by (8.3*b*) in which $A_3 = 0$ and B_3 and E_3 are given by (8.9). Such motion will only be possible if

$$\Gamma < \frac{2\sinh^2(ah)}{\sinh(ah)\cosh(ah) + ah}. \tag{8.11}$$

To complete the problem it is necessary to specify the value of Γ (or of a) and Φ. Guided by the pressure problem studied by Green & Naghdi (1986), we choose

$$c^2 a/g = \Gamma = \tanh(ah). \tag{8.12}$$

Again, following Naghdi & Rubin (1986), for the boat problem on water of small depth, we choose Φ so that the resultant of all the forces in (8.7) is normal to the plate. Then, we have

$$\Phi = \frac{\rho^* g l^2 F^4}{4}\left\{\frac{\sinh(ah)\cosh(ah) - ah}{\Gamma\sinh^2(ah)}\right\}\tan^2\alpha \tag{8.13}$$

and

$$\left.\begin{aligned}\mathscr{D}_R &= \frac{\rho^* g l^2 F^4}{2}\left\{\frac{\sinh(ah)\cosh(ah) - ah}{\Gamma\sinh^2(ah)}\right\}\tan^2\alpha,\\ \mathscr{L}_R &= \frac{\rho^* g l^2 F^4}{2}\left\{\frac{\sinh(ah)\cosh(ah) - ah}{\Gamma\sinh^2(ah)}\right\}\tan\alpha.\end{aligned}\right\} \tag{8.14}$$

Given (8.12) and (8.13) the lift L in (8.10) normal to the boat becomes

$$L = \frac{\rho^* g l^2[\sinh(ah)\cosh(ah) - ah]\sin\alpha}{4\sinh(ah)\cosh(ah)} + O(\tan^2\alpha). \tag{8.15}$$

The total resultant lift normal to the boat bottom is

$$L_T = L + \mathscr{D}_R \sin\alpha + \mathscr{L}_R \cos\alpha = \frac{\rho^* g l^2 [\sinh(ah)\cosh(ah) - ah][1 + 2(F^4/\Gamma^2)\sec^2\alpha]\sin\alpha}{4\sinh(ah)\cosh(ah)}. \tag{8.16}$$

When the angle is small the wave motion in region *III* is given approximately by

$$\phi - h = \beta = -\{a^{-1}\sin a(x - x_{II}) + l\cos\alpha\cos a(x - x_{II})\}\tan\alpha, \tag{8.17}$$

where a is determined by (8.12).

Two limiting cases of the foregoing analysis are of interest. When the depth of fluid is very large then $h \to \infty, \Gamma \to 1$ and the results (8.14) reduce to $(6.14)_2$ and $(6.13)_3$ whilst (8.16) reduces to the major term in (6.15). On the other hand when the depth is small, h is small, and (8.14) and (8.15) become

$$\mathscr{D}_R = \frac{\rho^* c^2 h}{3}\tan^2\alpha, \quad \mathscr{L}_R = \frac{\rho^* c^2 h}{3}\tan\alpha, \quad L = \frac{\rho^* g l^2}{2}\left(1 - \frac{c^2}{gh}\right)\sin\alpha. \tag{8.18}$$

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Appendix

Consider a finite three-dimensional body embedded in a euclidean three-space and identify each material point of the body by a convected system of coordinates θ^i, $(i = 1, 2, 3)$. Let $\boldsymbol{r}^*$ be the position vector, from a fixed origin, of a typical particle in the present configuration of the body at time t. Then,

$$\left.\begin{aligned} &\boldsymbol{r}^* = \boldsymbol{r}^*(\theta^i, t), \quad \boldsymbol{g}_i = \partial\boldsymbol{r}^*/\partial\theta^i, \quad \boldsymbol{g}^i\cdot\boldsymbol{g}_j = \delta^i_j, \\ &g_{ij} = \boldsymbol{g}_i\cdot\boldsymbol{g}_j, \quad g^{ij} = \boldsymbol{g}^i\cdot\boldsymbol{g}^j, \quad g^{\frac{1}{2}} = [\boldsymbol{g}_1\boldsymbol{g}_2\boldsymbol{g}_3], \end{aligned}\right\} \tag{A 1}$$

where $\boldsymbol{g}_i, \boldsymbol{g}^i$ are covariant and contravariant base vectors, respectively, g_{ij} and g^{ij} are covariant and contravariant metric tensors, respectively, and δ^i_j is the Kronecker delta. The velocity vector of a typical particle of the body denoted by $\boldsymbol{v}^*$ is defined by

$$\boldsymbol{v}^* = \dot{\boldsymbol{r}}^*, \tag{A 2}$$

where a superposed dot stands for the material time derivative holding θ^i fixed. For convenience we adopt the notation $\theta^3 = z$. We assume that the body is bounded by the surfaces

$$z = z_2(\theta^1, \theta^2), \quad z = z_1(\theta^1, \theta^2), \quad z_1 < z_2 \tag{A 3}$$

which are smooth and non-interesting and that, in this region, the position vector $\boldsymbol{r}^*$ is represented by

$$\left.\begin{aligned} &\boldsymbol{r}^* = \sum_{N=0}^{K} \lambda_N(z)\,\boldsymbol{d}_N, \quad \lambda_0 = 1, \quad \boldsymbol{d}_0 = \boldsymbol{r}, \\ &\boldsymbol{r} = \boldsymbol{r}(\theta^\alpha, t), \quad \boldsymbol{d}_N = \boldsymbol{d}_N(\theta^\alpha, t). \end{aligned}\right\} \tag{A 4}$$

The position vector $\boldsymbol{r}$ may be regarded as representing points on a two-dimensional surface whose base vectors and metric tensors are

$$\left.\begin{aligned} &\boldsymbol{a}_\alpha = \boldsymbol{a}_\alpha(\theta^\beta, t) = \partial\boldsymbol{r}/\partial\theta^\alpha, \quad \boldsymbol{a}^\alpha\cdot\boldsymbol{a}_\beta = \delta^\alpha_\beta, \quad \boldsymbol{a}_3 a^{\frac{1}{2}} = \boldsymbol{a}_1\times\boldsymbol{a}_2, \\ &a_{\alpha\beta} = \boldsymbol{a}_\alpha\cdot\boldsymbol{a}_\beta, \quad a^{\alpha\beta} = \boldsymbol{a}^\alpha\cdot\boldsymbol{a}^\beta, \quad a^{\frac{1}{2}} = [\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3], \end{aligned}\right\} \tag{A 5}$$

where Greek indices have the values 1,2. The particle velocity in the three-dimensional theory is then given by

$$\left.\begin{aligned} \boldsymbol{v}^* &= \sum_{N=0}^{K} \lambda_N(z)\,\boldsymbol{w}_N, \quad \boldsymbol{v} = \boldsymbol{w}_0, \\ \boldsymbol{v} &= \boldsymbol{v}(\theta^\alpha, t) = \dot{\boldsymbol{r}}, \quad \boldsymbol{w}_N = \boldsymbol{w}_N(\theta^\alpha, t) = \dot{\boldsymbol{d}}_N. \end{aligned}\right\} \tag{A 6}$$

Now let $\zeta^i (i = 1, 2, 3)$ be a system of fixed curvilinear coordinates in the same euclidean three-space and let points in this space be specified by a position vector $\bar{\boldsymbol{r}}^* = \bar{\boldsymbol{r}}^*(\zeta^i)$, with corresponding base vectors and metric tensors

$$\left.\begin{aligned} &\bar{\boldsymbol{g}}_i = \partial \bar{\boldsymbol{r}}^*/\partial \zeta^i, \quad \bar{\boldsymbol{g}}^i \cdot \bar{\boldsymbol{g}}_k = \delta^i_k, \quad \bar{g}_{ik} = \bar{\boldsymbol{g}}_i \cdot \bar{\boldsymbol{g}}_k, \\ &\bar{g}^{ik} = \bar{\boldsymbol{g}}^i \cdot \bar{\boldsymbol{g}}^k, \quad \bar{g}^{\frac{1}{2}} = [\bar{\boldsymbol{g}}_1 \bar{\boldsymbol{g}}_2 \bar{\boldsymbol{g}}_3]. \end{aligned}\right\} \tag{A 7}$$

We select a fixed reference surface in this space which, with its corresponding base vectors and metric tensors, is specified by

$$\left.\begin{aligned} &\bar{\boldsymbol{r}} = \bar{\boldsymbol{r}}(\zeta^\alpha), \quad \bar{\boldsymbol{a}}_\alpha = \partial \bar{\boldsymbol{r}}/\partial \zeta^\alpha, \quad \bar{\boldsymbol{a}}^\alpha \cdot \bar{\boldsymbol{a}}_\beta = \delta^\alpha_\beta, \quad \bar{\boldsymbol{a}}_3 \bar{a}^{\frac{1}{2}} = \bar{\boldsymbol{a}}_1 \times \bar{\boldsymbol{a}}_2, \\ &\bar{a}_{\alpha\beta} = \bar{\boldsymbol{a}}_\alpha \cdot \bar{\boldsymbol{a}}_\beta, \quad \bar{a}^{\alpha\beta} = \bar{\boldsymbol{a}}^\alpha \cdot \bar{\boldsymbol{a}}^\beta, \quad \bar{a}^{\frac{1}{2}} = [\bar{\boldsymbol{a}}_1 \bar{\boldsymbol{a}}_2 \bar{\boldsymbol{a}}_3]. \end{aligned}\right\} \tag{A 8}$$

Let $\bar{P}$ be the two-dimensional region of an arbitrary fixed surface bounded by a closed curve $\partial\bar{P}$ on the fixed surface (A 8), whose unit outward normal in the surface is

$$\bar{\boldsymbol{\nu}} = \bar{\nu}_\alpha \bar{\boldsymbol{a}}^\alpha = \bar{\nu}^\alpha \bar{\boldsymbol{a}}_\alpha. \tag{A 9}$$

In terms of the fixed coordinates ζ^i, the velocity of a particle at time t may be represented by

$$\left.\begin{aligned} &\boldsymbol{v}^* = \bar{\boldsymbol{v}}^*(\zeta^i, t) = \bar{v}^{*i} \bar{\boldsymbol{g}}_i = \sum_{N=0}^{K} \bar{\lambda}_N(\zeta)\,\bar{\boldsymbol{w}}_N, \\ &\zeta = \zeta^3, \quad \bar{\lambda}_0 = 1, \quad \bar{\boldsymbol{v}} = \bar{\boldsymbol{v}}(\zeta^\alpha, t) = \bar{\boldsymbol{w}}_0, \; \bar{\boldsymbol{w}}_N = \bar{\boldsymbol{w}}_N(\zeta^\alpha, t). \end{aligned}\right\} \tag{A 10}$$

The surfaces (A 3) which bound the body are now specified as

$$\zeta = \zeta_1(\zeta^1, \zeta^2, t), \quad \zeta = \zeta_2(\zeta^1, \zeta^2, t). \tag{A 11}$$

These surfaces are material surfaces and move with the body so that

$$\left.\begin{aligned} \partial\zeta_1/\partial t &= \bar{v}^{*3} - \bar{v}^{*\alpha}\,\partial\zeta_1/\partial\zeta^\alpha \quad (\zeta = \zeta_1), \\ \partial\zeta_2/\partial t &= \bar{v}^{*3} - \bar{v}^{*\alpha}\,\partial\zeta_2/\partial\zeta^\alpha \quad (\zeta = \zeta_2). \end{aligned}\right\} \tag{A 12}$$

Any function F^* associated with the body may be expressed in terms of θ^i, t or ζ^i, t. Thus,

$$F^*(\theta^i, t) = \bar{F}^*(\zeta^i, t) \tag{A 13}$$

and, in particular, for the mass density we write

$$\rho^*(\theta^i, t) = \bar{\rho}^*(\zeta^i, t). \tag{A 14}$$

We note that

$$\left.\begin{aligned} \dot{\rho}^* + \rho^* \operatorname{div} \boldsymbol{v}^* &= \partial\bar{\rho}^*/\partial t + \operatorname{div}(\bar{\rho}^* \bar{\boldsymbol{v}}^*) \\ &= \partial\bar{\rho}^*/\partial t + \bar{g}^{-\frac{1}{2}}(\bar{g}^{\frac{1}{2}} \bar{\rho}^* \bar{v}^{*i})_{,i}, \\ \dot{\overline{\rho^* F^*}} + \rho^* F^* \operatorname{div} \boldsymbol{v}^* &= (\partial/\partial t)\,(\bar{\rho}^* \bar{F}^*) + \operatorname{div}(\bar{\rho}^* \bar{F}^* \bar{\boldsymbol{v}}^*) \\ &= (\partial/\partial t)\,(\bar{\rho}^* \bar{F}^*) + \bar{g}^{-\frac{1}{2}}(\bar{g}^{\frac{1}{2}} \bar{\rho}^* \bar{F}^* \bar{v}^{*i})_{,i}, \end{aligned}\right\} \tag{A 15}$$

where $(\;)_{,i} = \partial(\;)/\partial\zeta^i$.

We now choose the convected coordinates θ^i such that at time t, the θ^i-curves coincide with the fixed ζ^i-curves, i.e. $\theta^i = \zeta^i + b^i$ where b^i are constants. Also, the moving surface represented by the vector $\boldsymbol{r}$ in (A 4) coincides with the fixed reference surface (A 8) at time t. With this choice of θ^i,

$$\left.\begin{aligned} \boldsymbol{g}_i = \bar{\boldsymbol{g}}_i, \quad g_{ij} = \bar{g}_{ij}, \quad g^{\frac{1}{2}} = \bar{g}^{\frac{1}{2}}, \quad \boldsymbol{a}_\alpha = \bar{\boldsymbol{a}}_\alpha, \quad a_{\alpha\beta} = \bar{a}_{\alpha\beta}, \text{ etc.} \\ \lambda_N(z) = \bar{\lambda}_N(\zeta). \end{aligned}\right\} \tag{A 16}$$

From (A 15) we have

$$(\dot{\rho}^* + \rho^* \operatorname{div} \boldsymbol{v}^*)\, \lambda_N(z)\, \lambda_M(z) = \frac{\partial}{\partial t}(\bar{\rho}^* \bar{\lambda}_N \bar{\lambda}_M) + \bar{g}^{-\frac{1}{2}}(\bar{g}^{\frac{1}{2}} \bar{\rho}^* \bar{\lambda}_N \bar{\lambda}_M \bar{v}^{*i})_{,i} - \bar{\rho}^* (\bar{\lambda}'_N \bar{\lambda}_M + \bar{\lambda}_N \bar{\lambda}'_M)\, \bar{v}^{*3},$$

or

$$\overline{\rho^* g^{\frac{1}{2}} \lambda_N \lambda_M}^{\,\boldsymbol{\cdot}} = \frac{\partial}{\partial t}(\bar{g}^{\frac{1}{2}} \bar{\rho}^* \bar{\lambda}_N \bar{\lambda}_M) + (\bar{g}^{\frac{1}{2}} \bar{\rho}^* \bar{\lambda}_N \bar{\lambda}_M \bar{v}^{*i})_{,i} - \bar{g}^{\frac{1}{2}} \bar{\rho}^* (\bar{\lambda}'_N \bar{\lambda}_M + \bar{\lambda}_N \bar{\lambda}'_M)\, \bar{v}^{*3},$$

where a prime denotes differentiation with respect to ζ. Hence,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{z_1}^{z_2} g^{\frac{1}{2}} \rho^* \lambda_N \lambda_M \,\mathrm{d}z = \frac{\partial}{\partial t} \int_{\zeta_1}^{\zeta_2} \bar{g}^{\frac{1}{2}} \bar{\rho}^* \bar{\lambda}_N \bar{\lambda}_M \,\mathrm{d}\zeta \\ + \frac{\partial}{\partial \zeta^\alpha} \int_{\zeta_1}^{\zeta_2} \bar{g}^{\frac{1}{2}} \bar{\rho}^* \bar{\lambda}_N \bar{\lambda}_M \bar{v}^{*\alpha} \,\mathrm{d}\zeta - \int_{\zeta_1}^{\zeta_2} \bar{g}^{\frac{1}{2}} \bar{\rho}^* (\bar{\lambda}'_N \bar{\lambda}_M + \bar{\lambda}_N \bar{\lambda}'_M)\, v^{*3} \,\mathrm{d}\zeta, \end{aligned}$$

where use is made of the surface conditions (A 12), and this may be put into the form

$$\overline{a^{\frac{1}{2}} \rho y_{MN}}^{\,\boldsymbol{\cdot}} = \frac{\partial}{\partial t}(\bar{a}^{\frac{1}{2}} \bar{\rho} \bar{y}_{MN}) + \frac{\partial}{\partial \zeta^\alpha}(\bar{a}^{\frac{1}{2}} \bar{\rho} \bar{\boldsymbol{v}}_{MN} \cdot \bar{\boldsymbol{a}}^\alpha) - \bar{a}^{\frac{1}{2}} \bar{\rho} (\bar{\boldsymbol{v}}_{MN} + \bar{\boldsymbol{v}}_{NM}) \cdot \bar{\boldsymbol{a}}_3, \tag{A 17}$$

where

$$\left.\begin{aligned} a^{\frac{1}{2}} \rho y_{MN} = \int_{z_1}^{z_2} g^{\frac{1}{2}} \rho^* \lambda_N \lambda_M \,\mathrm{d}z, \quad \bar{a}^{\frac{1}{2}} \bar{\rho} \bar{y}_{MN} = \int_{\zeta_1}^{\zeta_2} \bar{g}^{\frac{1}{2}} \bar{\rho}^* \bar{\lambda}_N \bar{\lambda}_M \,\mathrm{d}\zeta, \\ \bar{a}^{\frac{1}{2}} \bar{\rho} \bar{\boldsymbol{v}}_{MN} = \bar{\boldsymbol{a}}_\alpha \int_{\zeta_1}^{\zeta_2} \bar{g}^{\frac{1}{2}} \bar{\rho}^* \bar{\lambda}_M \bar{\lambda}_N \bar{v}^{*\alpha} \,\mathrm{d}\zeta + \bar{\boldsymbol{a}}_3 \int_{\zeta_1}^{\zeta_2} \bar{g}^{\frac{1}{2}} \bar{\rho}^* \bar{\lambda}'_M \bar{\lambda}_N \bar{v}^{*3} \,\mathrm{d}\zeta. \end{aligned}\right\} \tag{A 18}$$

Similarly, after multiplying $(\text{A } 15)_2$ by λ_N and using the surface conditions $(\text{A } 12)_1$, we obtain

$$\overline{a^{\frac{1}{2}} \rho F_N}^{\,\boldsymbol{\cdot}} = \overline{a^{\frac{1}{2}} \rho \sum_{M=0}^{K} y_{MN} f_M}^{\,\boldsymbol{\cdot}} = \frac{\partial}{\partial t}\left(\bar{a}^{\frac{1}{2}} \bar{\rho} \sum_{M=0}^{K} \bar{y}_{MN} \bar{f}_M\right) + \frac{\partial}{\partial \zeta^\alpha}\left(\bar{a}^{\frac{1}{2}} \bar{\rho} \sum_{M=0}^{K} \bar{f}_M \bar{\boldsymbol{v}}_{MN} \cdot \bar{\boldsymbol{a}}^\alpha\right) - \bar{a}^{\frac{1}{2}} \bar{\rho} \sum_{M=0}^{K} \bar{f}_M \bar{\boldsymbol{v}}_{MN} \cdot \bar{\boldsymbol{a}}_3, \tag{A 19}$$

where

$$F^* = \sum_{M=0}^{K} f_M \lambda_M, \quad \bar{F}^* = \sum_{M=0}^{K} \bar{f}_M \bar{\lambda}_M, \quad a^{\frac{1}{2}} \rho F_N = \int_{z_1}^{z_2} g^{\frac{1}{2}} \rho^* F^* \lambda_N \,\mathrm{d}z. \tag{A 20}$$

Let $\mathscr{P}$ be an arbitrary part of the moving surface $\boldsymbol{r} = \boldsymbol{r}(\theta^\alpha, t)$ which coincides with the part $\bar{P}$ of the fixed surface at time t, where $\bar{P}$ is an arbitrary part of the fixed surface (A 8). Then, in view of (A 19),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{P}} \rho F_N \,\mathrm{d}\sigma &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{P}} \rho \sum_{M=0}^{K} y_{MN} f_M \,\mathrm{d}\sigma \\ &= \frac{\partial}{\partial t} \int_{\bar{P}} \bar{\rho} \sum_{M=0}^{K} \bar{y}_{MN} \bar{f}_M \,\mathrm{d}\bar{\sigma} + \int_{\partial \bar{P}} \bar{\rho} \sum_{M=0}^{K} \bar{f}_M \bar{\boldsymbol{v}}_{MN} \cdot \bar{\boldsymbol{\nu}} \,\mathrm{d}\bar{s} - \int_{\bar{P}} \bar{\rho} \sum_{M=0}^{K} \bar{f}_M \bar{\boldsymbol{v}}_{NM} \cdot \mathrm{d}\bar{\boldsymbol{\sigma}}, \end{aligned} \tag{A 21}$$

where $\mathrm{d}\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{a}}_3\,\mathrm{d}\bar{\sigma}$. The results (A 19) and (A 21) are applied to particular cases in the main text:

$$(i)\quad F^* = \boldsymbol{v}^*,\quad \bar{F}^* = \bar{\boldsymbol{v}}^*,$$

where $\boldsymbol{v}^*, \bar{\boldsymbol{v}}^*$ have the representations (A 6) and (A 10) and

$$(ii)\quad F^* = \eta^* = \sum_{M=0}^{K} \eta_M \lambda_M,\quad \bar{F}^* = \bar{\eta}^* = \sum_{M=0}^{K} \bar{\eta}_M \bar{\lambda}_M,$$

where η^* is entropy density. In connection with entropy we record one other representation defined by

$$a^{\frac{1}{2}}\rho\tilde{\eta}_N = \int_{z_1}^{z_2} g^{\frac{1}{2}}\rho^*\eta^*\lambda_N\,\mathrm{d}z = a^{\frac{1}{2}}\rho \sum_{M=0}^{K} y_{MN}\eta_M. \tag{A 22}$$

Similarly, a further form is needed for the internal energy, namely

$$\left.\begin{aligned} a^{\frac{1}{2}}\rho\epsilon &= \int_{z_1}^{z_2} g^{\frac{1}{2}}\rho^*\epsilon^*\,\mathrm{d}z = \sum_{M=0}^{K}\sum_{N=0}^{K} a^{\frac{1}{2}}\rho y_{MN}\epsilon_{MN},\\ \epsilon^* &= \sum_{M=0}^{K}\sum_{N=0}^{K}\epsilon_{MN}\lambda_M\lambda_N,\quad \bar{\epsilon}^* = \sum_{M=0}^{K}\sum_{N=0}^{K}\bar{\epsilon}_{MN}\bar{\lambda}_M\bar{\lambda}_N,\end{aligned}\right\} \tag{A 23}$$

so that

$$\overline{\dot{a^{\frac{1}{2}}\rho\epsilon}} = a^{\frac{1}{2}}\rho\,\overline{\sum_{M=0}^{K}\sum_{N=0}^{K} y_{MN}\dot{\epsilon}_{MN}} = \frac{\partial}{\partial t}\left(\bar{a}^{\frac{1}{2}}\bar{\rho}\sum_{M=0}^{K}\sum_{N=0}^{K}\bar{y}_{MN}\bar{\epsilon}_{MN}\right) + \frac{\partial}{\partial\zeta^{\alpha}}\left(\bar{a}^{\frac{1}{2}}\bar{\rho}\sum_{M=0}^{K}\sum_{N=0}^{K}\bar{\epsilon}_{MN}\bar{\boldsymbol{v}}_{MN}\cdot\bar{\boldsymbol{a}}^{\alpha}\right). \tag{A 24}$$

The foregoing analysis gives a direct connection between integral balances in lagrangian and eulerian forms, and hence direct connections between lagrangian and eulerian forms of field equations. These latter forms may be obtained by a more direct method. Using (A 13) we have

$$\rho^*\dot{F}^*(\theta^i, t) = \bar{\rho}^*\left(\frac{\partial\bar{F}}{\partial t} + \bar{v}^{*i}\frac{\partial\bar{F}^*}{\partial\zeta^i}\right). \tag{A 25}$$

If we multiply each side by $\lambda_N(z) = \bar{\lambda}_N(\zeta)$ and use (A 20) we have

$$\sum_{M=0}^{K}\rho^*\lambda_N(z)\,\lambda_M(z)\dot{f}_M = \sum_{M=0}^{K}\bar{\rho}^*\bar{\lambda}_N(\zeta)\,\bar{\lambda}_M(\zeta)\frac{\partial\bar{f}_M}{\partial t} + \sum_{M=0}^{K}\bar{\rho}^*\bar{\lambda}_N(\zeta)\,\bar{\lambda}_M(\zeta)\,\bar{v}^{*\alpha}\frac{\partial\bar{f}_M}{\partial\zeta^{\alpha}} + \sum_{M=0}^{K}\rho^*\bar{\lambda}'_M(\zeta)\,\bar{\lambda}_N(\zeta)\,\bar{v}^{*3}\bar{f}_M. \tag{A 26}$$

We integrate (A 26) with respect to z between the limits z_1, z_2 or with respect to ζ between the limits ζ_1 and ζ_2 to obtain

$$\rho\sum_{M=0}^{K} y_{MN}\dot{f}_M = \bar{\rho}\sum_{M=0}^{K}\left\{\bar{y}_{MN}\frac{\partial\bar{f}_M}{\partial t} + (\bar{\boldsymbol{v}}_{MN}\cdot\bar{\boldsymbol{a}}^{\alpha})\frac{\partial\bar{f}_M}{\partial\zeta^{\alpha}} + \bar{f}_M\bar{\boldsymbol{v}}_{MN}\cdot\bar{\boldsymbol{a}}_3\right\}, \tag{A 27}$$

where we have also used (A 18).

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